Local smoothing for the Euclidean wave equation

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Abstract

We prove Sogge's local smoothing conjecture for the Euclidean wave equation in every spatial dimension at least two. We deduce the Bochner–Riesz conjecture, maximal estimates for these means, and sharp bounds for Stein's square function. We obtain Fourier restriction estimates for the sphere, the compact paraboloid, and the truncated cone throughout the interior of the conjectured range, as well as local smoothing for the Schrödinger equation. We also prove the Kakeya and Nikodym maximal conjectures; in particular, every Euclidean Kakeya set has full Hausdorff dimension.

1 Introduction

We prove Sogge's local smoothing conjecture for the Euclidean wave equation in every spatial dimension $n \geq 2$. A wave arriving from a thin spherical shell may have a much larger L^p norm at the instant when its rays come to a focus than it had initially. Sogge asked how many derivatives of the initial data are needed to control the norm averaged over a time interval [72, Section 4, Equation (4.5)].

Write $U(T) = e^{-\mathrm{i}T|D|}$ for the half-wave propagator, with $|D| = \sqrt{-\Delta}$. Put $I = [1, 2]$, $J = (1 - \Delta)^{1/2}$, and

$$s_+(p) = \max\left\{0, \frac{n-1}{2} - \frac{n}{p}\right\}.$$

Theorem 1.1 (Euclidean local smoothing)**.** *For $n \geq 2$, $2 \leq p < \infty$ and $\varepsilon > 0$,*

$$\|U(T)f\|_{L^p(\mathbb{R}^n \times I)} \leq C_{n,p,\varepsilon} \|J^{s_+(p)+\varepsilon} f\|_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

At $p = 2$ one can take $\varepsilon = 0$.

At a fixed nonzero time the sharp Sobolev order is $(n-1)(1/2 - 1/p)$ [54, 65, 68]. Thus for $p \geq p_{\mathrm{c}} = 2n/(n-1)$, averaging in time gains every amount less than $1/p$ over the fixed-time estimate. For $2 < p < p_{\mathrm{c}}$, arbitrarily small positive regularity suffices. A focusing wave shows that $s_+(p)$ is the smallest possible threshold (Proposition 8.1). We state the wave estimates for complex Schwartz data; density gives the indicated extensions, and conjugation gives the other sign of the half wave. Unqualified L^p norms of the data are spatial norms.

Contents

Sections 2–7 prove Theorem 1.1; Sections 9–12 give its applications.

1 Introduction . . . **1**
- 1.1 Consequences for Fourier analysis and geometry . . . 3
- 1.2 Development of the local smoothing problem . . . 3
- 1.3 An estimate on one frequency annulus . . . 4
- 1.4 A conserved current and two circles . . . 4

2 A conserved current for a superposition of waves . . . **6**
- 2.1 The symbol determines the Gaussian . . . 6
- 2.2 Two auxiliary flows . . . 7
- 2.3 Coordinates adapted to the wave . . . 8
- 2.4 The time component on a frequency cap . . . 10

3 The determinant and integration by parts . . . **10**
- 3.1 The Hessian in real and complex coordinates . . . 11
- 3.2 The Moore cofactor . . . 11
- 3.3 Column divergence and the boundary term . . . 12

4 The determinant energy and its boundary flux . . . **14**
- 4.1 The geometry of the fibers . . . 14
- 4.2 The Dirichlet potential . . . 15
- 4.3 Cancelling the full determinant . . . 15
- 4.4 Size of the energy on a fiber . . . 17

5 Estimating the energy on a fixed fiber . . . **18**

6 Propagation across a thin cylinder . . . **21**

7 Passage to Euclidean space and Sobolev estimates . . . **24**
- 7.1 Removing the period . . . 24
- 7.2 Interpolation and summation . . . 24

8 Two obstructions to endpoint smoothing . . . **26**

9 Spherical extension and Kakeya tubes . . . **27**
- 9.1 Global extension from the sphere and the truncated cone . . . 28
- 9.2 From caps to tubes . . . 29
- 9.3 The maximal estimate and Hausdorff dimension . . . 30
- 9.4 Two consequences for other families of segments . . . 31

10 Bochner–Riesz means . . . **32**

11 Stein's square function . . . **33**

12 The paraboloid and Schrödinger evolution . . . **36**

A The Dirichlet problem on a periodic fiber . . . **38**

B The sharp logarithm with angular regularity . . . **41**
- B.1 Observation on a ball . . . 42
- B.2 The polar kernel and the two endpoint norms . . . 43
- B.3 Summation over the spatial and angular frequencies . . . 44

C Disjoint packets and the weak spacetime norm . . . **46**

D A linear Dirichlet problem for the planar wave . . . **48**

E Curved characteristic sheets of symmetric systems . . . **49**
- E.1 Eigenvectors and the spatial fiber . . . 50
- E.2 The current and its Gaussian multiplier . . . 51
- E.3 The fixed cap estimate . . . 52
- E.4 A lower bound on a single patch . . . 53

References . . . **54**

1.1 Consequences for Fourier analysis and geometry

Bochner–Riesz means and Stein's square function. The Bochner–Riesz means $B_R^\alpha = (1-|D|^2/R^2)_+^\alpha$ are uniformly bounded on L^p for $1 \le p \le \infty$ and every $\alpha > \max\{0, n|1/p-1/2|-1/2\}$, as conjectured. For $2 \le p < \infty$ the same orders give a strong L^p bound for the supremum over all radii (Theorem 10.1). Stein's square function measures their variation with radius [73]. Keeping this variation inside a square integral gives its conjectured bounds for $2 \le p < \infty$ and $\alpha > 1/2 + s_+(p)$ (Theorem 11.1). The connection with the wave is that averaging in time against an oscillatory weight selects a thin shell in frequency. Sogge used this observation to pass from local smoothing to summability of Fourier integrals [72, Equations (4.5)–(4.6)]; for the reductions at fixed and variable radii see [8, Propositions 39 and 40].

Fourier restriction and Schrödinger waves. Give an incoming spherical wave an amplitude that varies with direction. Near the focus, rescaling recovers the Fourier extension of this amplitude from the sphere. We obtain the full interior range of global Fourier extension estimates $L^a \to L^b$: for $1 < a \le \infty$, $b < \infty$, it is given by

$$b > \frac{2n}{n-1}, \qquad b > \frac{n+1}{n-1}\,a', \qquad \frac{1}{a} + \frac{1}{a'} = 1.$$

Slicing gives the same bounds for the truncated cone in $\mathbb{R}^{n+1}$ (Corollary 9.3). Rescaling a small cap gives the compact paraboloid estimate (12.1) and local smoothing for Schrödinger waves in $n-1$ spatial dimensions, with the exact Sobolev order above their critical exponent (Corollary 12.1).

Kakeya and Nikodym sets. Random signs on disjoint frequency caps turn the square of the extension into a bound for overlaps of tubes. This proves the Kakeya maximal conjecture: the operator that takes f to the largest average of $|f|$ on a unit δ-tube in each direction has norm $O_\varepsilon(\delta^{-\varepsilon})$ from $L^n(\mathbb{R}^n)$ to $L^n(S^{n-1})$ for every $\varepsilon > 0$. In particular, every subset of $\mathbb{R}^n$ containing a unit line segment in each direction has Hausdorff dimension n (Corollary 9.1). A projective change of variables gives the corresponding Nikodym maximal estimate for tubes through a given point (see (9.7)).

1.2 Development of the local smoothing problem

The question grew out of Bourgain's theorem on maximal averages over circles [12]. At high frequency the Fourier transform of a circle of radius T has two leading terms, with phases $T|\xi|$ and $-T|\xi|$. This explains why a spacetime estimate for waves can help bound the supremum of circular averages over radii. Sogge used this connection to prove a positive smoothing gain for every $p > 2$ in two space dimensions [72, Theorem 4.1]. Mockenhaupt, Seeger and Sogge gave a proof of the circular maximal theorem using angular square functions [55], and developed local smoothing for Fourier integral operators under a curvature condition on the associated family of cones [56].

A persistent difficulty is to put waves moving in different directions back together. Wolff decomposed a neighborhood of the cone into thin angular sectors and controlled their sum by induction on scales. This gave every gain below $1/p$ for $p > 74$ in the plane [80]; Łaba and Wolff extended the argument to higher dimensions [49]. Bourgain and Demeter's cone decoupling theorem gives this gain for $p \ge 2(n+1)/(n-1)$ in every dimension [14, Theorem 1.2]. A different argument using radial Fourier multipliers attained the exact gain $1/p$, with no additional Sobolev loss, for $n \ge 4$ and $p > 2(n-1)/(n-3)$ [37, Theorem 1.2]. In the plane, Guth, Wang and Zhang reached the critical exponent $p = 4$ by controlling the L^4 norm of a wave by the spacetime square function of its angular pieces [35, Theorems 1.1 and 1.2]. For the canonical sectors of a truncated cone, Cushman,

Demeter and Wu give the sharp power $1/4$ of the logarithm in this square function inequality [20, Theorem 1.4]. To obtain a bound in terms of $\|f\|_4$, one must also estimate the square function from the initial data.

In higher dimensions the distinction between a partial gain at p_c and a full gain above p_c is substantial. At $n = p = 3$, Gao, Liu, Miao and Xi obtained every gain below $2/9$ by broad Fourier restriction estimates, compared with the conjectured $1/3$ [29, Corollary 1.3]. Gan, He, Li and Wu proved the full gain up to an arbitrarily small loss for $p \geq 10/3$ in $\mathbb{R}^3$ and on compact Riemannian three-manifolds [28, Theorem 1.3]; see also Wang's study of wave packet density [78]. Interpolating the $p = 10/3$ estimate with the L^2 identity gives every gain below $1/4$ at $p = 3$. Some three-dimensional metrics require $p \geq 10/3$ for gains arbitrarily close to $1/p$ [53], whereas the Euclidean exponent is 3.

1.3 An estimate on one frequency annulus

The proof is built around a single estimate at $p = p_c$. It quantifies the growth with frequency that the arbitrarily small Sobolev loss in Theorem 1.1 allows. Our Fourier convention is $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-iX\cdot\xi} f(X)\,dX$, so $U(T)$ has phase $X \cdot \xi - T|\xi|$ and inverse Fourier factor $(2\pi)^{-n}$.

Theorem 1.2 (Critical annular estimate)**.** *Let $n \geq 2$, $p = p_c$, and $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$. For $N \geq 2$ set $P_N = \varphi(D/N)$. Then*

$$\|U(T)P_N f\|_{L^p(\mathbb{R}^n \times I)} \leq C_{n,\varphi}[\log(2+N)]^{2/p}\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

Interpolation and summation over annuli yield Theorem 1.1 (Section 7). For unrestricted data on a full annulus the optimal logarithmic power in (1.2) lies between $1/p_c$ and $2/p_c$. The lower bound comes from the focusing example: every dyadic distance from the focus between one wavelength and unit distance contributes comparable mass to the p_c-th power of the spacetime norm (Proposition 8.1). With any positive angular regularity, the power $1/p_c$ suffices; Appendix B proves a sharper result with a logarithmic angular weight. For unrestricted data the lower bound persists even for the weak spacetime norm (Proposition 8.2).

1.4 A conserved current and two circles

Consider a finite sum of plane waves with $|\xi| \asymp N$ whose directions lie in one small cap. We need to take a square *after* summing these waves: summing the squares would discard the interference that produces a focus. An imaginary translation along a light ray filters the sum by direction. Translation by $-i\ell(1,\omega)$, with $|\omega| = 1$ and $\ell > 0$, multiplies a mode of frequency ξ by $\exp[-\ell|\xi|\,|\omega - \xi/|\xi||^2/2]$. Its Gaussian width in direction is therefore of order $(N\ell)^{-1/2}$ for $N\ell \gg 1$. The symmetric first-order wave system provides these light rays as its quadratic fluxes. The sum of two such fluxes gives the imaginary displacement. Two flows built from the continued wave then give a positive semidefinite tensor. Each of its real columns has zero divergence, and its time component still detects the original wave on the chosen cap (Section 2).

Here is the estimate that the construction must deliver. Set $h = N^{-1}$, let $a = Kh$ for a fixed large K, and write $p = p_c$. In addition to the physical time T, the construction has two auxiliary time coordinates $\tau \in \mathbb{R}^2$. Write $H(T,s)$ for the circular mean at $|\tau| = s$ of the nonnegative energy $\mathcal{F}(T,\tau)$ constructed below. For h small and nonzero cap data on a fixed spatial torus normalized by $\|f\|_2 = 1$, we prove

$$\begin{aligned}\int_1^2 \|U(T)f\|_p^p\,dT &\lesssim a^{-2}\int_1^2 H(T,2a)\,dT \\ &\lesssim [\log(2/h)]^2 a^{-2} H(0,4a) \lesssim [\log(2/h)]^2\|f\|_p^p.\end{aligned}$$

The first and last inequalities compare the energy with the wave. The middle one uses only a scalar differential inequality.

To obtain the energy, fix (T, τ) and let the remaining real and auxiliary spatial variables range over a bounded fiber with periodic real directions. Write $\mu > 0$ for the time component of the regularized current. There is a complex scalar Φ constant along the current. With $\lambda > 0$ large, take on each fiber the solution with positive definite spatial Hessian $H_{\mathrm{fib}}(\lambda|\Phi|^2 + u)$ of

$$\det H_{\mathrm{fib}}(\lambda|\Phi|^2 + u) = \lambda\mu, \qquad u|_\partial = 0.$$

Here H_{fib} is an n-by-n quaternionic Hermitian Hessian; its determinant on positive matrices is the product of the eigenvalues. The background costs nothing in the conservation law: since the current is tangent to the levels of Φ, its pairing with $H(|\Phi|^2)$ vanishes. In the spatial determinant, however, it supplies one direction of curvature. The Hessian of $\lambda|\Phi|^2$ has just one nonzero eigenvalue, of size λ. Adding αI to this background changes its determinant to $(\beta + \alpha)\alpha^{n-1}$, where $\beta \asymp \lambda$. Thus density $\mu \geq m$ on a small ball forces the depth of u at its center to be at least a constant times the squared radius times $m^{1/(n-1)}$, in the range of the construction. The integral $\int(-u)\mu$ detects the power $m^{n/(n-1)}$. Since μ recovers a square of the wave, the resulting exponent is $2n/(n-1)$.

A correction to $\int(-u)\mu$ makes the full determinant cancel when one differentiates the energy in (T, τ), while leaving the corrected energy comparable to that integral. The correction uses the integral of the spatial determinant along the segment of potentials from $\lambda|\Phi|^2$ to $\lambda|\Phi|^2 + u$, as in the classical Aubin–Mabuchi energy [5, 52]. The two integrations by parts use conservation of the wave current and of the Hessian cofactor (Sections 3 and 4). The cofactor contributes no boundary term: its normal–normal coefficient vanishes. Only the wave flux through the side of the fiber can give a positive contribution to the Laplacian of the energy. A comparison function made from a spatial average of the squared wave bounds $\mathcal{F}$ on the initial circle and the normal derivative of u at the side of the fiber. On the part of the boundary that carries wave flux, the Gaussian decay is stronger than the cost of recovering the wave in the interior. The resulting scalar inequality is

$$\Delta_{T,\tau}\mathcal{F} \leq Cs^{-2}e^{-cs/h}\mathcal{F}, \qquad 2a \leq s = |\tau| \leq 3 \quad \text{(Section 5)}.$$

It remains to compare the circles. The scalar inequality carries a lower bound at the observed times $1 \leq T \leq 2$ from $|\tau| = 2a$ to $|\tau| = 1$. At that fixed radius the mean inequality carries it to initial time; a second annulus comparison then takes it back to $|\tau| = 4a$. A radial harmonic function in the plane is affine in the logarithm of its radius. Each passage thus loses an inverse logarithm. Both losses can occur for a scalar supersolution even with zero potential (Remark 6.2). Taking a pth root and summing a fixed number of caps gives the claimed logarithmic bound on the torus. Section 7 removes the spatial period to prove Theorem 1.2, then sums the frequencies for the Sobolev estimate.

Related constructions. The bilinear flux of a symmetric hyperbolic system is classical [25]. Analytic continuation in an imaginary spacetime direction has also been used to study waves [42]. For positive-energy Klein–Gordon waves, Kaiser found a vector conserved in physical spacetime at each fixed imaginary displacement. Integration over displacements gives a current with nonnegative time density [41, Section 4.5, Equations (36)–(40)]. Here the continued wave gives a positive symmetric tensor whose columns have zero divergence in the combined physical and auxiliary variables.

Serre's compensated integrability estimates pair a positive divergence-free tensor with the Hessian of a real Monge–Ampère potential to bound a power of its determinant [69, Section 5.1]. The usual stress tensor of a wave need not be positive in $n \geq 2$ spatial dimensions [70, Section 4.1]. Our lifted tensor is positive semidefinite, but before averaging it is a sum of four real squares and

its determinant vanishes. We use its regularized time component as the prescribed density in a quaternionic determinant equation on each spatial fiber (Section 4).

The main proof ends in Section 7; Appendix A supplies the Dirichlet theorem used in the construction. Sections 9–12 give the applications. Appendix D replaces the determinant equation by a linear Dirichlet problem on a disk in two dimensions. Appendix E extends the current construction to a simple extremal eigenvalue of a real symmetric n-by-n pencil in n spatial variables whose transverse Hessian is definite, and gives an example with an irreducible cubic characteristic cone.

2 A conserved current for a superposition of waves

We use the flux of a symmetric first-order system twice. It selects an imaginary displacement that gives each mode a Gaussian weight; applied to the continued wave, it gives a positive conserved tensor. The modes are summed before taking the square, so their interference remains in the time component of the tensor. We shall apply the construction on a fixed spatial torus: an annulus then contains only finitely many modes, and integration by parts has no boundary in the real spatial variables. Section 7 transfers the estimate to $\mathbb{R}^n$.

2.1 The symbol determines the Gaussian

Choose an orthonormal frame $X = (x, b) \in \mathbb{R}^d \times \mathbb{R}$, where $d = n - 1$, and put

$$t = (T + b)/\sqrt{2}, \qquad s_0 = (b - T)/\sqrt{2}.$$

In these null coordinates the wave operator is $2\partial_t\partial_{s_0} + \Delta_x$. At a fixed frequency $\kappa > 0$ in s_0 this is the free Schrödinger operator $2\mathrm{i}\kappa\partial_t + \Delta_x$ in the transverse variables. A mode with physical phase $X \cdot \xi - T|\xi|$ and $\xi_b + |\xi| > 0$ takes the form

$$u_{\kappa,v} = c_{\kappa,v} \exp\Big(\mathrm{i}\kappa \left(s_0 + v \cdot x - \tfrac{1}{2}|v|^2 t\right)\Big), \qquad \kappa = \frac{\xi_b + |\xi|}{\sqrt{2}}, \qquad v = \frac{\xi_x}{\kappa}. \tag{2.1}$$

Here i denotes the ordinary Fourier imaginary unit. Throughout the construction $u_{\mathrm{w}} = \sum_{\kappa,v} u_{\kappa,v}$ is a finite sum with $\kappa > 0$ and complex coefficients.

Write E_{ab} for the elementary $n \times n$ matrix with its nonzero entry in position (a, b), indexing the states from 0 to d. For the real spacetime coordinates $\mathsf{X} = (t, x, s_0)$ take the symmetric matrices

$$C^t = 2E_{00}, \qquad C^{x_j} = E_{0j} + E_{j0}, \qquad C^{s_0} = -\sum_{j=1}^{d} E_{jj}, \qquad \mathscr{P} = \sum_{\mu} C^\mu \partial_{\mathsf{X}^\mu}.$$

Indeed, $\mathscr{P}(\partial_{s_0}\psi, \nabla_x\psi)^{\mathsf{t}}$ has first entry $(2\partial_t\partial_{s_0} + \Delta_x)\psi$ and all other entries zero. At the covector $p = \kappa(-|v|^2/2, v, 1)$ of (2.1), the symbol $C(p) = \sum_\mu p_\mu C^\mu$ satisfies

$$-C(p) = \kappa \begin{pmatrix} |v|^2 & -v^{\mathsf{t}} \\ -v & I_d \end{pmatrix}, \qquad -w^{\mathsf{t}} C(p) w = \kappa |w' - v w_0|^2. \tag{2.2}$$

In particular the kernel is spanned by $E(v) = (1, v)^{\mathsf{t}}$. This gives a vector solution $f_{\mathrm{sys}} = \sum_{\kappa,v} \kappa^{d/2} E(v) u_{\kappa,v}$ of $\mathscr{P} f_{\mathrm{sys}} = 0$.

To see the Gaussian in physical coordinates, translate (T, X) by $-\mathrm{i}\rho(1, \omega)$ with $\rho > 0$ and $|\omega| = 1$. A mode of frequency ξ acquires the factor

$$e^{\rho(\xi\cdot\omega - |\xi|)} = e^{-\rho|\xi|\,|\widehat{\xi} - \omega|^2/2}, \qquad \widehat{\xi} = \xi/|\xi|.$$

For $|\xi| \asymp N$ and $\rho N \gg 1$, its angular width is $(\rho N)^{-1/2}$. The system itself supplies these light rays. For a real state $w = (w_0, w')$ its flux $q(w)^\mu = w^{\mathsf{t}} C^\mu w/2$ is $(w_0^2, w_0 w', -|w'|^2/2)$ in null coordinates. In physical coordinates this gives $q(w)^T = |q(w)^X| = (w_0^2 + |w'|^2/2)/\sqrt{2}$; at the polarization $E(v)$ the ray is $q(E(v)) = (|\xi|, \xi)/\kappa$. Take the sum of the fluxes of two real states,

$$q^\mu = q(w_1)^\mu + q(w_2)^\mu = \tfrac{1}{2} \sum_{\ell=1}^{2} w_\ell^{\mathsf{t}} C^\mu w_\ell, \qquad w_1, w_2 \in \mathbb{R}^n. \tag{2.3}$$

At $\mathsf{X} - \mathrm{i}q$ the weight is $e^{p\cdot q}$. Pairing corresponding entries of the states, write

$$z = w_1 + \mathbf{k} w_2 = (R, \zeta) \in \mathbb{C}_{\mathbf{k}} \times \mathbb{C}_{\mathbf{k}}^d, \qquad \mathbb{C}_{\mathbf{k}} = \{a + \mathbf{k} b : a, b \in \mathbb{R}\}, \quad \mathbf{k}^2 = -1.$$

Thus $q^\mu = z^* C^\mu z/2$, with * the complex conjugate transpose. Adding the two real squares in (2.2) gives

$$e^{p\cdot q} = \exp\Big(-\tfrac{\kappa}{2} |\zeta - v R|^2\Big). \tag{2.4}$$

This explains the normalization in f_{sys}: the square of $\kappa^{d/2} e^{p\cdot q}$ integrates to π^d in ζ for every R.

2.2 Two auxiliary flows

Recall the bilinear conservation identity for symmetric systems, as in Friedrichs [25]. For constant real symmetric matrices C^μ, the flux of states a, b is $j^\mu(a, b) = a^{\mathsf{t}} C^\mu b$. Its divergence for two fields is $(\mathscr{P} a)^{\mathsf{t}} b + a^{\mathsf{t}} \mathscr{P} b$. The derivative of q^μ in the direction δw_ℓ is exactly $j^\mu(w_\ell, \delta w_\ell)$. This derivative gives the physical component of the following vector fields.

Lemma 2.1. *Let $\mathscr{P} = \sum_\mu C^\mu \partial_{\mathsf{X}^\mu}$ have constant real symmetric coefficients. For a holomorphic solution $\mathscr{P} f_{\mathrm{sys}} = 0$ and the real displacement (2.3), put, wherever defined,*

$$G = \overline{f_{\mathrm{sys}}(\mathsf{X} - \mathrm{i}q)}, \qquad V_{\ell a} = \sum_\mu (C^\mu w_\ell)_a \, \partial_{\mathsf{X}^\mu}, \qquad D_{\ell a} = \partial_{w_{\ell a}} + \mathrm{i} V_{\ell a}, \qquad \mathcal{Z}_\ell = \sum_a G_a D_{\ell a}.$$

In the real coordinates (X, w_1, w_2), the tensor

$$B = \mathrm{Re} \sum_{\ell=1}^{2} \mathcal{Z}_\ell \otimes \overline{\mathcal{Z}_\ell} \tag{2.5}$$

is symmetric and positive semidefinite, and each of its columns has zero divergence.

Proof. The fields $D_{\ell a}$ are the complex vector fields associated with the tube coordinates $\mathsf{X} - \mathrm{i}q$ as in [6, Section 1]. Each annihilates every component of $\mathsf{X} - \mathrm{i}q$, hence $D_{\ell a} \overline{G} = 0$. Conjugation gives $\partial_{w_{\ell a}} G = \mathrm{i} V_{\ell a} G$. Each $D_{\ell a}$ has zero divergence, so

$$\operatorname{div} \mathcal{Z}_\ell = 2\mathrm{i} \sum_a V_{\ell a} G_a = 2\mathrm{i}\, w_\ell^{\mathsf{t}} \mathscr{P} G = 0.$$

For the ordinary flat connection, the identity $D_{\ell a} \overline{G} = 0$ leaves only the derivatives of the coefficients of $\overline{D_{mb}}$ when computing the acceleration:

$$\nabla_{\mathcal{Z}_\ell} \overline{\mathcal{Z}_m} = -\mathrm{i}\, \delta_{\ell m} \sum_\mu (G^{\mathsf{t}} C^\mu \overline{G}) \partial_{\mathsf{X}^\mu}.$$

The vector in the sum is real by symmetry of the matrices. The product rule for $\operatorname{div}(\mathcal{Z}_\ell \otimes \overline{\mathcal{Z}_\ell})$ now gives zero after taking real parts. Finally, if $\mathcal{Z}_\ell = A_\ell + \mathrm{i} B_\ell$ with real fields, its contribution to (2.5) is $A_\ell \otimes A_\ell + B_\ell \otimes B_\ell$. □

This cancellation also identifies the physical current in the lifted variables: the imaginary part of the last differentiation identity at $m = \ell$ gives $[A_\ell, B_\ell] = \sum_\mu (G^{\mathsf{t}} C^\mu \overline{G}) \partial_{\mathsf{X}^\mu}$. For the estimate we use the four real squares in the conserved tensor B. We must express them in coordinates attached to spacetime.

2.3 Coordinates adapted to the wave

We want one auxiliary pair for each of the $n+1$ real spacetime coordinates. In the target, write each triple as $\mathsf{X}^\mu + \omega_\mu \mathbf{i}$, with $\omega_\mu \in \mathbb{C}_\mathbf{k}$. The quaternions $\mathbb{H}$ have units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k}$, $\mathbf{ki} = \mathbf{j}$. Thus the target triples lie in $\mathcal{D} = \mathrm{span}_\mathbb{R}(1, \mathbf{i}, \mathbf{j})$. We identify the Fourier complex plane with $\mathbb{C}_\mathbf{i}$. Every quaternion is uniquely $\alpha + \mathbf{k}\beta$ with $\alpha, \beta \in \mathbb{C}_\mathbf{i}$; these two planes are orthogonal for the real scalar product $\langle u, v\rangle = \mathrm{Re}(u\overline{v})$. We use columns with scalars acting on the right; * on a matrix denotes quaternionic conjugate transpose.

The representation we seek for B is the following. For a quaternionic Hermitian matrix A on spacetime indices, set

$$A^\mathbb{R}_{\mu\alpha,\nu\beta} = \mathrm{Re}(\overline{e_\alpha} A_{\mu\nu} e_\beta), \qquad (e_0, e_1, e_2) = (1, \mathbf{i}, \mathbf{j}). \tag{2.6}$$

If $A = \chi\chi^*$, this real tensor is the Gram matrix of the rows $\overline{e_\alpha}\chi_\mu$. In other words the three rows over each μ have the form $Q, -\mathbf{i}Q, -\mathbf{j}Q$.

In source coordinates, write the four squares of (2.5) as a Gram matrix by setting $Q_I = \mathcal{Z}^I_1 + \mathbf{k}\mathcal{Z}^I_2$ for each coordinate I. Orthogonality of $\mathbb{C}_\mathbf{i}$ and $\mathbf{k}\mathbb{C}_\mathbf{i}$ gives $B^{IJ} = \mathrm{Re}(Q_I \overline{Q_J})$. With $M_{\mu a} = (C^\mu z)_a$, the rows in the real spacetime and state directions are

$$Q_{\mathsf{X}^\mu} = \sum_a M_{\mu a} \mathbf{i} G_a, \qquad (Q_{w_{1a}}, Q_{w_{2a}}) = (G_a, \mathbf{k}G_a).$$

We now choose the target auxiliary coordinates to give the desired rows $-\mathbf{i}Q_{\mathsf{X}^\mu}, -\mathbf{j}Q_{\mathsf{X}^\mu}$. For a holomorphic coordinate $\omega_\mu(z)$ let $L_{\mu a} = \partial_{z_a}\omega_\mu$. Multiplication by $c = c_1 + \mathbf{k}c_2$ on an auxiliary pair has real rows $(c_1, -c_2), (c_2, c_1)$, so it sends $(G_a, \mathbf{k}G_a)$ to $(\bar{c}G_a, \mathbf{k}\bar{c}G_a)$. Consequently the new pair over μ starts with $\sum_a \overline{L_{\mu a}} G_a$. Taking $L = M$ gives exactly $-\mathbf{i}Q_{\mathsf{X}^\mu}$, since $-\mathbf{i}c\,\mathbf{i} = \bar{c}$ for $c \in \mathbb{C}_\mathbf{k}$. Its partner is $\mathbf{k}(-\mathbf{i}Q_{\mathsf{X}^\mu}) = -\mathbf{j}Q_{\mathsf{X}^\mu}$.

The holomorphic flux map. The derivative $L_{\mu a} = (C^\mu z)_a$ integrates to the quadratic $z^\mathsf{t} C^\mu z/2$, by symmetry. Thus the Hermitian square $z^* C^\mu z/2$ weights the modes, while the bilinear square transforms their Gram rows. There are n complex state coordinates and $n+1$ spacetime coordinates. Introduce a passive $\eta \in \mathbb{C}_\mathbf{k}$, in whose two real directions the flows have zero component. For any constant real symmetric system of Lemma 2.1 with these dimensions and any fixed $c \in \mathbb{R}^{n+1}$, set

$$\Theta : (z, \eta) \longmapsto \omega, \qquad \omega_\mu = \tfrac{1}{2} z^\mathsf{t} C^\mu z + c^\mu \eta.$$

The map fixes the real spacetime coordinates. On a local inverse chart its real volume Jacobian is $J = |\det_\mathbb{C}[M, c]|^2$, where $M_{\mu a} = (C^\mu z)_a$. Push the tensor forward as a density:

$$\widetilde{B} \circ \Theta = J^{-1} D\Theta\, B\, D\Theta^\mathsf{t}. \tag{2.7}$$

The two differentials transform the Gram rows as above, and J^{-1} converts source volume to target volume. Hence

$$\widetilde{B} = (\widetilde{\chi}\widetilde{\chi}^*)^\mathbb{R}, \qquad \widetilde{\chi}_\mu \circ \Theta = J^{-1/2} \sum_a M_{\mu a} \mathbf{i} G_a. \tag{2.8}$$

As a function of ω, the passive coordinate satisfies $(\partial_\omega \eta) M = 0$: varying z at fixed η gives the columns of M. Thus $(\partial_\omega \eta)\widetilde{\chi} = 0$.

To verify conservation, test the Kth column against the gradient of a real smooth function φ compactly supported in the target chart. Changing variables in (2.7), the integral for the weak divergence becomes $-\sum_{I,L} \int B^{IL} (\partial_L \Theta^K) \partial_I (\varphi \circ \Theta)$. Integration by parts in source coordinates gives

$$J(\mathrm{div}\, \widetilde{B})^K \circ \Theta = \sum_L (\partial_L \Theta^K)(\mathrm{div}\, B)^L + \sum_{I,L} B^{IL} \partial_I \partial_L \Theta^K.$$

The first sum vanishes by Lemma 2.1. In the second sum the two diagonal state blocks of B are both $(\mathrm{Re}(G_a\overline{G_b}))_{ab}$, and the mixed state block is zero. Each real component of the holomorphic map satisfies $\partial_{w_{1a}w_{1b}}\Theta^K + \partial_{w_{2a}w_{2b}}\Theta^K = 0$. The remaining coordinates enter Θ affinely, so the second sum vanishes as well.

The wave in these coordinates. For the wave matrices, add η in the s_0 coordinate. In the null frame the holomorphic flux map becomes

$$\Theta : (R, \zeta, \eta) \longmapsto (Y, P, S_1) = (R^2, R\zeta, \eta - \zeta\cdot\zeta/2). \tag{2.9}$$

If (τ, P, Z) denote the target auxiliary coordinates in the physical frame (T, x, b), then $Y = (\tau+Z)/\sqrt{2}$ and $S_1 = (Z-\tau)/\sqrt{2}$. Inverting (2.9) on $Y \neq 0$ gives $R = \sqrt{Y}$ on a local branch and $\zeta = P/R$. The useful single-valued coordinates are the ratio of the transverse states to R and the passive variable:

$$W = \frac{P}{Y} = \frac{\zeta}{R}, \qquad \Phi = S_1 + \frac{P\cdot P}{2Y} = \eta, \qquad y = |Y|. \tag{2.10}$$

All dots between complex vectors here are bilinear; $|\cdot|$ denotes the Euclidean norm. The complex Jacobian determinant of Θ is $2R^{d+1}$, so its real volume Jacobian is

$$J = 4|R|^{2d+2}. \tag{2.11}$$

Under the derivative M in the state directions a polarization vector $E(v)$ maps to

$$ME(v) = (2R, \zeta + Rv, -\zeta\cdot v)^{\mathsf{t}} = 2RV^v, \qquad V^v = \big(1, (v+W)/2, -W\cdot v/2\big)^{\mathsf{t}}. \tag{2.12}$$

In particular Φ is constant in each such direction because it pulls back to the passive variable. Explicitly, put $\ell(w) = (-w\cdot w/2, w, 1)$; thus $\kappa\ell(v)$ is the real phase covector. One checks $\ell(w)V^v = -(w-W)\cdot(w-v)/2$. The column lies in the kernels of both $\ell(v)$ and $\partial\Phi = \ell(W)$. Differentiating $W = P/Y$ gives the second identity in

$$(\partial\Phi)V^v = 0, \qquad (\partial W)V^v = \frac{v-W}{2Y}. \tag{2.13}$$

These are complex differentials in (Y, P, S_1). The current stays on a level set of Φ, while each mode changes W in proportion to its angular mismatch $v - W$.

For $e = Y/y$ put $\iota_e(\alpha + \mathrm{i}\beta) = \alpha + e\mathbf{i}\beta$. This is an isometric embedding of the Fourier complex numbers into $\mathbb{H}$: $\mathbf{i}c = \bar{c}\mathbf{i}$ for $c \in \mathbb{C}_{\mathbf{k}}$, so $(e\mathbf{i})^2 = -1$. Define the positive weight

$$m_{\kappa,v} = (\kappa/y)^{d/2}\exp\Big(-\tfrac{\kappa y}{2}|W-v|^2\Big).$$

The factor $y^{-d/2}$ comes from the change of volume.

Proposition 2.2 (The wave current)**.** *For any finite superposition* (2.1)*, set*

$$\chi = \sum_{\kappa,v} V^v\,\iota_e(\overline{u_{\kappa,v}})\,m_{\kappa,v}, \qquad A^0 = \chi\chi^*. \tag{2.14}$$

On $Y \neq 0$ the matrix A^0 is smooth and positive semidefinite, $(\partial\Phi)A^0 = 0$, and each column of $(A^0)^{\mathbb{R}}$ has zero real divergence. These conclusions hold also after the orthogonal change from null to physical coordinates. In particular,

$$V^v_T = (1 + W\cdot v/2)/\sqrt{2}, \qquad A^0_{TT} = |\chi_T|^2.$$

Proof. We identify the column in (2.8) on one inverse chart. By (2.4) and (2.12), the contribution of a single mode to $\widetilde{\chi}$ equals

$$V^v \, g\mathbf{i} \, \overline{u_{\kappa,v}} \, m_{\kappa,v}, \qquad g = R/|R|.$$

Indeed $|\zeta - vR|^2 = y|W - v|^2$ and $2|R|/\sqrt{J} = y^{-d/2}$. Multiply every row on the right by the same unit quaternion $(g\mathbf{i})^{-1}$; this preserves their Gram matrix. Conjugation by g takes $\mathbf{i}$ to $g^2\mathbf{i} = e\mathbf{i}$. Since $\mathbf{i}$ commutes with the Fourier amplitudes, the resulting column is precisely (2.14), with V^v on the left. The formula involves Y itself, hence is smooth and independent of the square root used in its derivation. Positivity follows from its outer product and tangency from (2.13). The change back to physical coordinates acts by the same real orthogonal matrix on each of the three coordinate slots, so it preserves the column divergence and the row–column pairing. Finally $T = (t - s_0)/\sqrt{2}$ gives the stated expression for V_T^v. □

For a single κ and real W, the exponential in $m_{\kappa,v}$ is a Gaussian window in transverse frequency κv, centered at κW. Since $V_t^v = 1$, the null component A_{tt}^0 is the squared modulus of the scalar wave observed through this window: up to normalization, its Husimi density [39]; see [45, Section 2.3]. For the full wave, (2.14) sums the amplitudes over κ before taking the square.

2.4 The time component on a frequency cap

At a fixed auxiliary point write $V_T^v = b_0 + \mathbf{k}b_1$, so that

$$b_0 = \frac{1 + \mathrm{Re}(W) \cdot v/2}{\sqrt{2}}, \qquad b_1 = \frac{\mathrm{Im}_{\mathbf{k}}(W) \cdot v}{2\sqrt{2}}.$$

Here $\mathrm{Im}_{\mathbf{k}}(\alpha + \mathbf{k}\beta) = \beta$. Let S_ℓ be the ordinary scalar spatial multiplier which sends each mode in (2.1) to $b_\ell m_{\kappa,v} u_{\kappa,v}$ at this auxiliary point. By (2.14),

$$\chi_T = \iota_e(\overline{\mathsf{S}_0 u_{\mathrm{w}}}) + \mathbf{k}\, \iota_e(\overline{\mathsf{S}_1 u_{\mathrm{w}}}), \qquad A_{TT}^0 = |\mathsf{S}_0 u_{\mathrm{w}}|^2 + |\mathsf{S}_1 u_{\mathrm{w}}|^2. \tag{2.15}$$

Indeed the two terms lie in the conjugates of the orthogonal planes $\mathbb{C}_{\mathbf{i}}$ and $\mathbf{k}\mathbb{C}_{\mathbf{i}}$ by a square root g of e in $\mathbb{C}_{\mathbf{k}}$. At $W = 0$ the second multiplier vanishes: $2A_{TT}^0$ is exactly the squared modulus of the wave with weight $(\kappa/y)^{d/2} \exp(-\kappa y|v|^2/2)$ on each mode.

For the estimates take $u_{\mathrm{w}} = e^{-\mathrm{i}T|D|} f$ on the fixed spatial torus $\mathbb{R}^n/(100\mathbb{Z}^n)$. We choose the axis b along the center of a frequency cap, leaving the torus itself fixed. For $0 < h \le 1$ suppose $\widehat{f}$ is supported in $c_0/h \le |\xi| \le C_0/h$, $|v| \le \gamma$, with fixed $0 < c_0 < C_0$ and a small aperture $\gamma > 0$ independent of h. The identity $|\xi| = \kappa(1 + |v|^2/2)/\sqrt{2}$ gives $\kappa \asymp h^{-1}$. At $y \asymp h$ and $|W|, |v| \le 1/10$, the Gaussian and b_0 are bounded below, so $h^d \mathsf{S}_0$ has a symbol bounded above and below on the cap. For larger y/h the cost of inverting it depends on the mismatch $|W - v|$. There are $O(h^{-n})$ frequencies in the annulus, uniformly in the frame; thus $\sum_{\kappa,v} |c_{\kappa,v}| \lesssim h^{-n/2}\|f\|_2$ by Cauchy–Schwarz and Parseval. The formulas in null coordinates may be used on the covering space: the resulting current is periodic in the original real spatial variables. We normalize nonzero cap data by $\|f\|_2 = 1$ in the energy estimates that follow.

3 The determinant and integration by parts

The wave current in Proposition 2.2 is positive and has divergence-free real columns. We shall pair it with a Hessian. The essential pairing is the ordinary real contraction $A : H(v) = A^{\mathbb{R}} : D^2 v$: each spacetime index still has its three real coordinates. The choice of Hessian has two further consequences. The squared modulus of a holomorphic function contributes at most one quaternionic direction. In n spatial variables, if its nonzero eigenvalue is β, adding ϵI gives determinant $(\beta + \epsilon)\epsilon^{n-1}$.

This is where the denominator in $p_c = 2n/(n-1)$ enters. The cofactor of the Hessian also has divergence-free columns; this allows us to integrate the determinant equation by parts, including at the boundary.

3.1 The Hessian in real and complex coordinates

Use the triples $z_a = x_a + P_a\mathbf{i} \in \mathcal{D}$ of Section 2, with $P_a \in \mathbb{C}_\mathbf{k}$. We allow a fourth real coordinate as well: Appendix A uses it to rotate the normal at the boundary. Thus for each of m coordinate indices let $z_a = \sum_{\alpha=0}^{k-1} e_\alpha z_{a\alpha}$, where $k = 3$ or 4, $(e_0, e_1, e_2, e_3) = (1, \mathbf{i}, \mathbf{j}, \mathbf{k})$, and the $z_{a\alpha}$ are real. For a real C^2 function u put $\partial_{a\alpha} = \partial/\partial z_{a\alpha}$ and define

$$G_a(u) = \sum_\alpha e_\alpha\, \partial_{a\alpha} u, \qquad H_{ab}(u) = \sum_{\alpha,\beta} e_\alpha \bar{e}_\beta\, \partial_{a\alpha}\partial_{b\beta} u. \tag{3.1}$$

Thus $G(u)$ is the ordinary gradient written as a quaternionic column. The symbols ∇, Δ and $\partial_\mathbf{n}$ denote the Euclidean gradient, Laplacian and outward normal derivative.

Lemma 3.1 (Hessian calculus)**.** *In either number of slots, $H(u)$ is Hermitian and $\operatorname{tr} H(u) = \Delta u$. The linear map from the real Hessian to $H(u)$ sends $\xi\xi^\mathsf{t}$ to gg^*, where $g_a = \sum_\alpha e_\alpha \xi_{a\alpha}$. It sends nonnegative real forms to nonnegative quaternionic forms. The product rule is*

$$H(uv) = uH(v) + vH(u) + G(u)G(v)^* + G(v)G(u)^*. \tag{3.2}$$

In four slots H is covariant under quaternionic unitary changes of coordinates, and in three slots under real orthogonal changes of the coordinate index applied in all slots.

If u depends only on $P \in \mathbb{C}_\mathbf{k}^m$, then

$$G_a(u) = 2(\partial_{\bar{P}_a} u)\mathbf{i}, \qquad H_{ab}(u) = 4\partial_{\bar{P}_a}\partial_{P_b} u. \tag{3.3}$$

In particular, a holomorphic map $F = (F_1, \dots, F_l)$ with complex Jacobian $J_{\nu a} = \partial_{P_a} F_\nu$ satisfies

$$H\Big(\sum_\nu |F_\nu|^2\Big) = 4J^*J. \tag{3.4}$$

For every smooth real function ρ of F, $G(\rho \circ F)$ belongs to the quaternionic image of J^*.

Proof. Interchanging the real derivatives gives Hermitian symmetry. In a diagonal entry the terms with $\alpha \neq \beta$ cancel in pairs; the remaining terms have coefficient one. Formula (3.1) applied to $\xi\xi^\mathsf{t}$ gives gg^*, and sums of these rank-one forms give every nonnegative real form. The same formula and the ordinary product rule give (3.2). In four slots the real transpose of a quaternionic unitary U acts on the encoded covector as U^*. Thus the Hessian rule for a real rank-one form gives $H(u \circ U) = U^*(H(u) \circ U)U$; linearity extends this to every real Hessian. Real orthogonal matrices on the coordinate index also preserve the three-slot subspace, giving the stated covariance there.

For $P_a = p_a + \mathbf{k}q_a$ we have $2\partial_{\bar{P}_a} = \partial_{p_a} + \mathbf{k}\partial_{q_a}$. Since $\mathbf{ki} = \mathbf{j}$, this gives the gradient in (3.3). The four terms in H_{ab} have coefficients $1, -\mathbf{k}, \mathbf{k}, 1$ in the order $(p_a p_b, p_a q_b, q_a p_b, q_a q_b)$, which gives its Hessian formula. The complex chain rule now gives (3.4) and $G(\rho \circ F) = 2J^*(\partial_{\bar{F}_\nu}\rho)_\nu\, \mathbf{i}$, with the derivatives of ρ evaluated at F. □

3.2 The Moore cofactor

For Hermitian quaternionic matrices use the real inner product $A : B = \operatorname{Re}\operatorname{tr}(AB)$. The real part of the trace is cyclic; in particular $A : (vv^*) = v^*Av$. Write $\det A$ for the Moore determinant, the

product of the m real eigenvalues counted with quaternionic multiplicity. These conventions agree with [2, Section 1.1, especially Theorems 1.1.8–1.1.9]. The next identities apply also to the full Hessian in the energy argument, whose spatial block is positive definite but whose time entry need not have a sign.

Lemma 3.2 (Matrix identities)**.** *The Moore determinant is a homogeneous polynomial of degree m. Its cofactor $C(A)$ is the Hermitian matrix defined by*

$$C(A) : B = \frac{d}{dt} \det(A + tB)\Big|_{t=0}. \tag{3.5}$$

It is polynomial of degree $m - 1$, and

$$C(A) = (\det A)A^{-1} \quad (A > 0). \tag{3.6}$$

If $A \geq 0$, then $C(A) \geq 0$; if also A is singular, $C(A)$ kills the image of A. On positive definite matrices $\log\det$ *is concave and increasing in the matrix order.*

For a unit vector $v \in \mathbb{H}^m$ let $A|_{v^\perp}$ denote the compression of the form to its quaternionic orthogonal complement. For every Hermitian A and real t,

$$v^* C(A) v = \det(A|_{v^\perp}), \qquad \det(A + tvv^*) = \det A + t\, v^* C(A) v. \tag{3.7}$$

The determinant on the zero-dimensional space is one. Finally, for a Hermitian $M = \left(\begin{smallmatrix} \ell & c^* \\ c & D \end{smallmatrix}\right)$ with $D > 0$, put $\sigma = \ell - c^* D^{-1} c$ and let v_0 be the first coordinate vector. Then

$$\det M = (\det D)\sigma, \qquad M - \sigma v_0 v_0^* \geq 0. \tag{3.8}$$

Proof. A Hermitian matrix has a quaternionic orthonormal eigenbasis: maximize $v^* A v$ on the unit sphere and repeat on the orthogonal complement. By Newton's identities its determinant is a polynomial in the power sums $\operatorname{Re}\operatorname{tr}(A^j)$, $1 \leq j \leq m$. This gives (3.5) with the stated degrees. In the real representation on $\mathbb{R}^{4m}$ each eigenvalue occurs four times. Thus for $A > 0$, $\log\det A$ is one fourth of the real log determinant. Differentiation proves (3.6), monotonicity and concavity. On a diagonal matrix each cofactor eigenvalue is the product of the other $m - 1$ eigenvalues, first for positive matrices by (3.6), then for all diagonal matrices by polynomiality. Unitary invariance proves the claims for singular A.

Completing the square shows that the form of $M - \sigma v_0 v_0^*$ at (b, w) is $(w + D^{-1}cb)^* D (w + D^{-1}cb)$. For $\sigma > 0$ the real Schur determinant formula gives $(\det M)^4 = (\det D)^4 \sigma^4$; take the positive fourth root. Both sides of $\det M = (\det D)\sigma$ are polynomials in ℓ, so the identity holds for every σ. In a basis with v first, adding tvv^* changes only ℓ. For positive A the change in its determinant is therefore exactly $t \det(A|_{v^\perp})$. Differentiation in t gives the first formula in (3.7); polynomiality extends both to every Hermitian A. For $m = 1$ they hold directly. □

3.3 Column divergence and the boundary term

In k slots the real tensor (2.6) of a Hermitian field $A(z)$ has coefficients

$$A^{\mathbb{R}}_{a\alpha, b\beta} = \operatorname{Re}(\bar{e}_\alpha A_{ab} e_\beta). \tag{3.9}$$

This symmetric real matrix represents the Hermitian form on the encoded real vectors: $(\nabla v)^{\mathsf{t}} A^{\mathbb{R}} \nabla u = \operatorname{Re}(G(v)^* A G(u))$. Likewise $A : H(u) = A^{\mathbb{R}} : D^2 u$, where the pairing on the right is the ordinary real matrix inner product. Consequently, whenever the real columns have zero divergence,

$$A : H(u) = \operatorname{div}(A^{\mathbb{R}} \nabla u). \tag{3.10}$$

One sufficient condition is the quaternionic row identity

$$\sum_{a,\alpha} \bar{e}_\alpha \, \partial_{a\alpha} A_{ab} = 0 \quad (1 \le b \le m) : \tag{3.11}$$

multiply on the right by e_β and take real parts. Also, if $AG(w) = 0$ for a real scalar function w, then wA preserves either form of column conservation already satisfied by A. Indeed the extra row in the product rule is $G(w)^* A = 0$.

Alesker showed that the real tensor representing the Moore cofactor of a positive definite quaternionic Hessian has divergence-free columns. This puts the linearized Monge–Ampère operator in divergence form [4, Lemma 4.6]. The following polynomial proof also covers the three slots used by the wave current and allows an indefinite or singular Hessian.

Lemma 3.3 (Cofactor divergence)**.** *For $k = 3$ or 4 and a smooth real u, the Hessian cofactor satisfies*

$$\sum_{a,\alpha} \bar{e}_\alpha \, \partial_{a\alpha} C(H(u))_{ab} = 0 \quad (1 \le b \le m). \tag{3.12}$$

Proof. For $m = 1$ the cofactor is one. Otherwise put $j_0 = m-1$. Polarize the cofactor: for Hermitian inputs $M_1, \dots, M_{j_0}$, denote the coefficient of $t_1 \cdots t_{j_0}$ in $C(\sum_j t_j M_j)$ by $Q(M_1, \dots, M_{j_0})$. This is multilinear, and on the diagonal equals $j_0! \, C(M_1)$.

Take $M_j = g_j g_j^*$. For $t_j > 0$, $\sum_j t_j g_j g_j^*$ is singular: its image is the quaternionic span of the j_0 columns. Its cofactor therefore kills each g_j, by Lemma 3.2. This polynomial identity in t holds coefficient by coefficient. Thus $Q = Q(g_1 g_1^*, \dots, g_{j_0} g_{j_0}^*)$ kills every g_j, and $g_j^* Q = 0$ as well since Q is Hermitian.

Test the polarized identity on $u_j(z) = \exp(\xi_j \cdot z)$ with arbitrary real covectors ξ_j, using the Euclidean dot product. Set $(g_j)_a = \sum_\alpha e_\alpha (\xi_j)_{a\alpha}$. The Hessian symbol in Lemma 3.1 gives $Q(H(u_1), \dots, H(u_{j_0})) = \exp((\sum_j \xi_j) \cdot z) \, Q$. After cancellation of this exponential, its row divergence is $(\sum_j g_j)^* Q = 0$. To pass to arbitrary smooth inputs, observe that the row divergence is a multilinear differential expression with constant coefficients: its value on these exponential functions, divided by their product, is the polynomial whose coefficients multiply the corresponding derivatives of the inputs. That polynomial vanishes for all covectors, so all coefficients vanish. Taking every input equal to u proves (3.12). □

Here is the boundary convention used for both the current and the cofactor. On a smooth domain $\Omega = \{r < 0\}$ assume $\nabla r \neq 0$ on $\partial\Omega$. Write $\mathbf{n} = \nabla r / |\nabla r|$ for the real outward unit normal and $\nu = G(r)/|\nabla r|$ for the same vector in quaternionic coordinates. By (3.10), Green's identity for a field with divergence-free real columns is

$$\int_\Omega (v \, A : H(u) - u \, A : H(v)) = \int_{\partial\Omega} \mathrm{Re}(v \, \nu^* A G(u) - u \, \nu^* A G(v)) \, \mathrm{d}S. \tag{3.13}$$

Volume and surface measure are Euclidean. The same formula holds in periodic coordinates, with test functions of compact support in any unbounded directions.

If u vanishes on $\partial\Omega$, then $G(u) = \nu \, \partial_{\mathbf{n}} u$ there. Also $u = \rho r$ for a smooth ρ near the boundary: use r as one local coordinate and integrate the derivative of u in that coordinate. The cross terms of (3.2) then vanish on compression, giving

$$H(u)|_{\nu^\perp} = \rho \, H(r)|_{\nu^\perp} \qquad \text{on } \partial\Omega. \tag{3.14}$$

In four slots this calculation gives the invariance of the quaternionic Levi form under a change of defining function [3]. Functions independent of the fourth slot give the same formula in three slots. In particular the normal flux coefficient $\nu^* C(H(u)) \nu$ depends only on this compressed form, by (3.7). This is why the determinant boundary term can be evaluated without knowing the second derivative of u in the real normal direction.

4 The determinant energy and its boundary flux

The time component of the wave current is a sum of squares. After regularization it gives the density for a spatial Dirichlet potential: a lower bound on the density throughout a ball forces a definite depth at its center. To compare that depth at different times, we need a scalar energy on the three real coordinates $z = (T, \tau) \in \mathbb{R} \times \mathbb{C}_{\mathbf{k}}$ whose Laplacian is bounded above by the wave flux through the side of the spatial fiber.

Keep the annular wave on the torus from Section 2, normalized by $\|f\|_2 = 1$. Put $w_0 = 1/10$ and assume its sector $|v| \le \gamma$ has $\gamma < w_0/4$. Write $s = |\tau|$ and $a = Kh$, where $K \ge 1$ is fixed and h is small enough that $4a \le 1$. All the constructions of this section take place on $a < s < 4$.

4.1 The geometry of the fibers

At fixed z, let X run over the spatial torus and let the imaginary spatial variables $(P, Z) \in \mathbb{C}_{\mathbf{k}}^d \times \mathbb{C}_{\mathbf{k}}$ vary in the domain

$$\Omega_z : \quad r = \frac{|W|^2}{w_0^2} + 100|Q|^2 - 1 < 0, \qquad Q = \Phi/a. \tag{4.1}$$

The shape of the fiber reflects two properties of the current from Proposition 2.2: it is tangent to the level sets of Φ, and its Gaussian is centered at $W = v$ for a mode of slope v. Only the W part of the wall normal can therefore carry wave flux. In (W, Q) the fibers are all the same ellipsoid. The definitions $Y = (\tau + Z)/\sqrt{2}$, $W = P/Y$ and $\Phi = (Z - \tau)/\sqrt{2} + P \cdot P/(2Y)$ give the inverse map

$$Y = \frac{\sqrt{2}\,\tau + aQ}{1 + W \cdot W/2}, \qquad P = WY, \qquad Z = \sqrt{2}\,Y - \tau. \tag{4.2}$$

The denominator is bounded away from zero on $|W| \le w_0$, and $|aQ| \le s/10$, so $y = |Y| \asymp s$. Both the chart and its inverse are holomorphic on neighborhoods of these closed fibers and depend smoothly on τ. On the total domain $\Omega = \{(z, X, P, Z) : a < s < 4,\ (X, P, Z) \in \Omega_z\}$ we use H, G for the full Hessian and gradient from Lemma 3.1. Their spatial parts at fixed z are $H_{\mathrm{fib}}, G_{\mathrm{fib}}$. Fiber integrals use ordinary volume $\mathrm{d}\zeta = \mathrm{d}X\ \mathrm{d}P\ \mathrm{d}Z$ in the original coordinates.

Lemma 4.1 (Fiber geometry)**.** *The spatial Hessian $H_{\mathrm{fib}}(r)$ is positive definite. The full $H(r)$ is positive semidefinite with a one-dimensional quaternionic kernel contained in both $G(r)^{\perp}$ and $\ker H(|\Phi|^2)$. On the closed fibers, uniformly for $a < s < 4$,*

$$H_{\mathrm{fib}}(r) \ge cs^{-2} I, \qquad G_{\mathrm{fib}}(r) G_{\mathrm{fib}}(r)^* \le H_{\mathrm{fib}}(r), \tag{4.3}$$

$$\mathrm{d}P\ \mathrm{d}Z \asymp a^2 s^{2d}\, \mathrm{d}W\ \mathrm{d}Q, \qquad \det H_{\mathrm{fib}}(r) \asymp a^{-2} s^{-2d}. \tag{4.4}$$

The spatial $H_{\mathrm{fib}}(|\Phi|^2)$ has rank one with nonzero eigenvalue comparable to one. Also $|G(|W|^2)| \lesssim s^{-1}$.

Proof. Set $F = (W/w_0, 10Q)$ and let J be its complex Jacobian in (τ, P, Z), with spatial submatrix J_{fib}. Since $r = |F|^2 - 1$, Lemma 3.1 gives

$$H(r) = 4J^*J, \qquad G(r) = 2J^*F\mathbf{i}.$$

At fixed τ the spatial Jacobian J_{fib} is invertible. In the full space we can vary τ while keeping (W, Q) fixed; the resulting direction spans $\ker J$ over $\mathbb{C}_{\mathbf{k}}$ and its quaternionic span is the null line of $H(r)$. This line is orthogonal to $G(r)$, which belongs to the image of J^*, and lies in $\ker H(|\Phi|^2)$ because the last row of J is proportional to $\partial\Phi$. By (4.2), derivatives of (P, Z) in W are $O(s)$ and those in Q are $O(a)$. Thus $\|J_{\mathrm{fib}}^{-1}\| \lesssim s$; together with $|F| \le 1$ this proves (4.3).

The complex Jacobian determinant of (4.2) in (W, Q) is $\sqrt{2}\, aY^d/(1 + W \cdot W/2)$: taking the exterior product of the differentials of $P = WY$ with $\mathrm{d}Y$ leaves just Y^d times the product of the differentials of W with $\mathrm{d}Y$. Squaring the absolute value gives the volume formula. The determinant of $4J_{\mathrm{fib}}^* J_{\mathrm{fib}}$ is $4^n |\det_{\mathbb{C}} J_{\mathrm{fib}}|^2$, giving the second bound in (4.4). Finally the spatial derivatives of Φ are bounded and $\partial_Z \Phi = (1 - W \cdot W/2)/\sqrt{2}$ is bounded away from zero. All first derivatives of $W = P/Y$ are $O(s^{-1})$. □

4.2 The Dirichlet potential

The time density can vanish. We first smooth the wave current in X and cut it off in the passive coordinate Q. To make its time density strictly positive, we then add a conserved current whose image is the null line of $H(r)$; it has zero flux through the wall. For $\ell > 0$ let J_ℓ be the periodization on the spatial torus of the probability kernel $c_n \ell^{-n}(1 + |X/\ell|^2)^{-n-2}$. Choose a smooth $0 \le g \le 1$ on $\mathbb{C}_{\mathbf{k}}$, equal to one on $|Q| \le 1/100$ and zero on $|Q| \ge 1/20$. Define

$$A_{\mathrm{w}} = g(Q)(J_h *_X A^0), \qquad A = A_{\mathrm{w}} + h^{50n} C(H(r)), \qquad \mu = A_{TT}. \tag{4.5}$$

Convolution takes place only in the real spatial variable. Since $\partial\Phi$ is independent of X, the averaged current still annihilates the image of $H(|\Phi|^2)$ and hence $G(g(Q))$. The product rule for divergence therefore preserves conservation under the cutoff. By Lemmas 3.2 and 3.3, the regularizing cofactor is positive semidefinite and divergence free. Its image is $\ker H(r)$, perpendicular to $G(r)$, and its TT entry is $\det H_{\mathrm{fib}}(r) > 0$. Thus

$$A \ge 0, \qquad \mu > 0, \qquad A : H(|\Phi|^2) = 0. \tag{4.6}$$

Put $B_0 = \lambda|\Phi|^2$, $B = H(B_0)$, $B_{\mathrm{fib}} = H_{\mathrm{fib}}(B_0)$, with $\lambda = h^{-30n}$. By (4.6), the full background satisfies $A : B = 0$. Its spatial block B_{fib} has one nonzero eigenvalue $\beta \asymp \lambda$. The factor h^{50n} keeps the added density below the wave estimates of Section 5; we take λ large enough that

$$\mu^{1/d} \le \lambda \tag{4.7}$$

for all sufficiently small h. To see this, each Gaussian amplitude factor in χ has size at most $(\kappa/y)^{d/2} \lesssim h^{-d}$. The columns V^v are bounded and $\sum |c_{\kappa,v}| \lesssim h^{-n/2}$ by the cap estimate in Section 2, so $\|A^0\| \lesssim h^{-n-2d}$. The regularizer adds only $O(h^{50n} a^{-2} s^{-2d})$ to μ by (4.4).

On each spatial fiber there is a unique smooth real u satisfying

$$D_u = H_{\mathrm{fib}}(B_0 + u) > 0, \qquad \det D_u = \lambda\mu, \qquad u|_{\partial\Omega_z} = 0. \tag{4.8}$$

Indeed, the spatial defining Hessian is positive definite, and $B_{\mathrm{fib}} \ge 0$ has zero determinant since $n \ge 2$. Together with $\mu > 0$ these are the hypotheses of Theorem A.1. The chart (4.2) and the parameter assertion of that theorem give smooth dependence on z up to the lateral boundary; its barriers (A.2) give $u \le 0$. To see how the equation measures depth, add αI to the spatial background. Then $\det(B_{\mathrm{fib}} + \alpha I) = (\beta + \alpha)\alpha^d$; thus for $0 < \alpha \le \lambda$ a density $\lambda\mu$ corresponds to an isotropic comparison quadratic of curvature $\alpha \asymp \mu^{1/d}$. Section 5 will use such quadratics on balls.

4.3 Cancelling the full determinant

The quantity that measures the depth of the potential against its density is $X_0(z) = \int_{\Omega_z} (-u)\mu \,\mathrm{d}\zeta$. Since the real columns of A have zero divergence, Green's formula for $\Delta_z X_0$ has interior term $-\int_{\Omega_z} A : H(u) \,\mathrm{d}\zeta$. In a related estimate for real positive tensors, Serre uses the determinant of a

Schur complement to isolate a diagonal entry of a second tensor [71, Proposition 2.2]. Here only the spatial block of $H(B_0+u)$ is known to be positive. Separate off its scalar Schur complement:

$$H(B_0+u)=\begin{pmatrix}\ell & c^*\\ c & D_u\end{pmatrix}=\sigma_0 e_T e_T^*+\Sigma,\qquad \sigma_0=\ell-c^*D_u^{-1}c,\qquad \Sigma=\begin{pmatrix}c^*D_u^{-1}c & c^*\\ c & D_u\end{pmatrix}\geq 0. \tag{4.9}$$

Here e_T is the time coordinate vector. Completing the square gives $\Sigma\geq 0$. In the contraction with A the background term $A:B$ vanishes by (4.6); in the determinant, $\det D_u=\lambda\mu$ gives $\det H(B_0+u)=\lambda\mu\sigma_0$. Thus

$$A:H(u)=A:\Sigma+\mu\sigma_0=A:\Sigma+\lambda^{-1}\det H(B_0+u). \tag{4.10}$$

The full determinant has no prescribed sign. We look for a second fiber integral whose Laplacian cancels this term. Write $D_v=H_{\mathrm{fib}}(B_0+v)$. Along the segment $B_0+\theta u$, the cofactor of the full Hessian has TT entry $\det D_{\theta u}$, and its contraction with $H(u)$ differentiates the full determinant in θ. This suggests integrating the spatial determinant along the segment. For a smooth real v on Ω_z with $D_v>0$ and zero boundary values, set

$$E_z(v)=\frac{1}{\lambda}\int_{\Omega_z} v\int_0^1 \det D_{\theta v}\,\mathrm{d}\theta\;\mathrm{d}\zeta,\qquad \mathcal{J}_z(v)=E_z(v)-\int_{\Omega_z} v\mu\,\mathrm{d}\zeta. \tag{4.11}$$

The segment integral is the quaternionic counterpart of the Aubin–Mabuchi energy: Mabuchi's functional L integrates the variation of a Kähler potential against its complex Monge–Ampère measure [52, (2.2.1)]. Here the determinant is a polynomial, so the integrand is smooth up to the singular endpoint $D_0=B_{\mathrm{fib}}$. Define $\mathcal{F}(z)=\mathcal{J}_z(u)=E_z(u)+X_0(z)$.

Proposition 4.2 (Energy balance)**.** *The energy $\mathcal{F}$ is smooth on $a<|\tau|<4$. Let ρ denote the boundary value of u/r, which extends smoothly from the interior. In the three variables $z=(T,\tau)$ one has*

$$\Delta_z\mathcal{F}(z)+\int_{\Omega_z} A:\Sigma\;\mathrm{d}\zeta=\int \rho\, G(r)^* A_{\mathrm{w}} G(r)\,\delta(r)\,\mathrm{d}\zeta. \tag{4.12}$$

The normal seen by the current is that of the moving wall in the total space, so $G(r)$ is the full *gradient. The integration is on one fiber: $\delta(r)\,\mathrm{d}\zeta=\mathrm{d}S_{\mathrm{fib}}/|\nabla_{\mathrm{fib}} r|$, where the spatial gradient is nonzero by the ellipsoid chart. Since $A:\Sigma\geq 0$, the boundary integral bounds $\Delta_z\mathcal{F}$ from above.*

Proof. Denote the boundary integral in (4.12) by $\mathcal{B}(z)$. The two identities whose sum gives the balance are

$$\begin{aligned}\Delta_z X_0&=\mathcal{B}(z)-\int_{\Omega_z} A:\Sigma\,\mathrm{d}\zeta-\lambda^{-1}\int_{\Omega_z}\det H(B_0+u)\,\mathrm{d}\zeta,\\ \Delta_z[E_z(u)]&=\lambda^{-1}\int_{\Omega_z}\det H(B_0+u)\,\mathrm{d}\zeta.\end{aligned}$$

For a fixed complex product domain, Åhag et al. [1, Theorem 3.1] identified the Laplacian of the slice energy, up to normalization, with the integral of the full Monge–Ampère measure over the fiber. Here the fiber moves, so the second identity requires a boundary calculation. Both follow by testing with a smooth $\eta=\eta(z)$ compactly supported in the parameter domain. The only entry of $H(\eta)$ is $H_{TT}(\eta)=\Delta_z\eta$. For any Hermitian matrix field M with divergence-free real columns, Green's formula (3.13) on the *total* domain gives

$$\int_\Omega u M_{TT}\Delta_z\eta=\int_\Omega \eta\, M:H(u)-\int_{\partial_{\mathrm{lat}}\Omega}\eta\,(\nu^* M\nu)\,\partial_{\mathbf{n}}u\,\mathrm{d}S. \tag{4.13}$$

Here $\partial_{\mathrm{lat}}\Omega$ is the side $r = 0$, and volume on Ω is $\mathrm{d}z\,\mathrm{d}\zeta$. The term containing $u\nabla\eta$ vanishes because $u = 0$ on the lateral boundary; there are no end terms because of the support of η. The unit outward normal $\mathbf{n} = \nabla r/|\nabla r|$ on the total domain has quaternionic column $\nu = G(r)/|\nabla r|$. Since r is a smooth defining function, $u = r\rho$ near the boundary with ρ smooth and $\rho \geq 0$ on $r = 0$. In particular $\partial_{\mathbf{n}} u = \rho|\nabla r|$.

For X_0, take $M = A$ in (4.13) and change sign since X_0 integrates $(-u)\mu$. The interior term follows from (4.10). On the wall the regularizer $C(H(r))$ has image perpendicular to ν, so only A_{w} contributes. Substituting $\partial_{\mathbf{n}} u = \rho|\nabla r|$ and $\mathrm{d}S = |\nabla r|\delta(r)\,\mathrm{d}z\,\mathrm{d}\zeta$ gives $\mathcal{B}(z)$.

For $E_z(u)$, take $M = C(H(B_0 + \theta u))$, whose columns are conserved by Lemma 3.3. Here the entire normal coefficient vanishes. Indeed (3.14) gives, on $r = 0$,

$$H(B_0 + \theta u)|_{\nu^\perp} = (B + \theta\rho H(r))|_{\nu^\perp}.$$

The null line of $H(r)$ belongs to both $\nu^\perp$ and $\ker B$ by Lemma 4.1; the restricted form is therefore singular. The cofactor restriction formula in Lemma 3.2 yields $\nu^* M \nu = 0$, independently of the sign of the full Hessian. The same formula in the e_T direction gives $M_{TT} = \det D_{\theta u}$. Now integrate (4.13) in θ and divide by λ. On the left we obtain $\int E_z(u)\Delta_z\eta\,\mathrm{d}z$; on the right, integration of $M : H(u) = \partial_\theta \det H(B_0 + \theta u)$ gives the full determinant at $\theta = 1$, since $\det B = 0$. This proves the second identity and hence the balance in distributions. The chart (4.2) makes the volume and surface integrals smooth in z, so it also holds pointwise. □

4.4 Size of the energy on a fiber

Fix z. The segment integral in (4.11) is a primitive of the spatial determinant: for a smooth variation w with zero boundary values, one has

$$\mathrm{d}\mathcal{J}_z(v)[w] = \int_{\Omega_z} w(\lambda^{-1}\det D_v - \mu)\,\mathrm{d}\zeta,$$
$$\mathrm{d}^2\mathcal{J}_z(v)[w,w] = -\lambda^{-1}\int_{\Omega_z} G_{\mathrm{fib}}(w)^* C(D_v) G_{\mathrm{fib}}(w)\,\mathrm{d}\zeta.$$

For the first identity, differentiating under the integral in (4.11) produces a term $\theta v C(D_{\theta v}) : H_{\mathrm{fib}}(w)$. By cofactor divergence one may interchange v and w in its fiber integral; both vanish on the boundary. The integrand in θ is then $w\,\partial_\theta(\theta \det D_{\theta v})$. Differentiating once more and integrating by parts gives the second formula. It is strictly negative for every nonzero w, since $C(D_v) > 0$. Thus the Dirichlet solution is the unique maximizer on this convex class. This is the smooth determinant variational principle with a rank-one background. For zero background, Wan proved the first variation and the corresponding variational characterization in quaternionic finite energy classes [76, Section 4].

Since $u \leq 0$ and $D_{\theta u} = (1-\theta)B_{\mathrm{fib}} + \theta D_u \geq 0$, the correction $E_z(u)$ is nonpositive. At a solution the linear term in $\mathcal{F}$ uses the determinant at $B_0 + u$. Thus $\lambda\mathcal{F}$ has the form of Aubin's J-functional [5, p. 145], with $B_0 + u$ as reference potential and B_0 as argument in the relative formula [10, Section 2.1]. The rank-one background lets us calculate how much of X_0 remains in $\mathcal{F} = X_0 + E_z(u)$.

Lemma 4.3 (Energy comparison)**.** *On every spatial fiber,*

$$\mathcal{F} = \frac{n-1}{n+1}X_0 + \frac{1}{\lambda n(n+1)}\int_{\Omega_z} G_{\mathrm{fib}}(u)^* C(D_u) G_{\mathrm{fib}}(u)\,\mathrm{d}\zeta. \tag{4.14}$$

In particular, $\frac{n-1}{n+1}X_0 \leq \mathcal{F} \leq \frac{n}{n+1}X_0$.

Proof. Because B_{fib} has rank one, the determinant along the segment has just two terms:

$$\det D_{\theta u} = \theta^n \det D_u + (1-\theta)\theta^{n-1} C(D_u) : B_{\mathrm{fib}}.$$

Write $I = \int_{\Omega_z} (-u) C(D_u) : B_{\mathrm{fib}} \, \mathrm{d}\zeta \geq 0$. Integrating in θ gives $\mathcal{F} = nX_0/(n+1) - I/[\lambda n(n+1)]$, in particular the upper bound. Also $C(D_u) : D_u = n \det D_u$, so

$$n\lambda X_0 - I = \int_{\Omega_z} (-u) C(D_u) : H_{\mathrm{fib}}(u) \, \mathrm{d}\zeta = \int_{\Omega_z} G_{\mathrm{fib}}(u)^* C(D_u) G_{\mathrm{fib}}(u) \, \mathrm{d}\zeta.$$

This last integration by parts uses $u = 0$ at the boundary. Substitution proves (4.14) and its lower bound. □

5 Estimating the energy on a fixed fiber

On a fiber at fixed (T, τ), the only positive contribution to the energy balance comes from the wave current crossing the spatial wall $r = 0$. The Gaussian is small where the current reaches the wall; comparison with a spatial average of the wave bounds the other factor, the boundary value of u/r. This gives an upper bound for the flux in terms of the wave norm. To bound it by the energy itself, we must recover that same norm from the interior. There we invert the Gaussian and compare u with a quadratic on each ball of radius h.

Proposition 5.1. *Fix an annulus $c_1 \leq h|\xi| \leq c_2$, with $0 < c_1 < c_2$. There are constants $\gamma > 0$, $K \geq 1$, and $c, C > 0$, depending only on n and the annulus, with the following property. Let $a = Kh$, $0 < h < 1/(4K)$, and let f be a trigonometric polynomial on the fixed torus, with $\|f\|_2 = 1$ and frequencies in this annulus and in $|v| \leq \gamma$ in some orthonormal frame. Write $g_T = e^{-iT|D|} f$ and let $\mathcal{F}(T, \tau)$ be its energy from Section 4. Then*

$$\mathcal{F}(0, \tau) \leq C a^2 \|f\|_{p_{\mathrm{c}}}^{p_{\mathrm{c}}} \qquad (|\tau| = 4a), \tag{5.1}$$

$$\mathcal{F}(T, \tau) \geq c a^2 \|g_T\|_{p_{\mathrm{c}}}^{p_{\mathrm{c}}} \qquad (|\tau| = 2a), \tag{5.2}$$

$$\Delta_z \mathcal{F}(T, \tau) \leq V(s) \mathcal{F}(T, \tau), \qquad V(s) = C s^{-2} e^{-cs/h} \qquad (2a \leq s = |\tau| \leq 3). \tag{5.3}$$

The last two inequalities hold for all $T \in \mathbb{R}$.

Proof. Write $d = n - 1$, $q = n/d = p_{\mathrm{c}}/2$, and $\sigma = s/h$. Throughout the proof $2a \leq s \leq 3$. On the fiber

$$r = |W|^2/w_0^2 + 100|Q|^2 - 1 < 0, \qquad w_0 = 1/10, \quad Q = \Phi/a,$$

we use two regions. On $r = 0$ the cutoff in the current vanishes unless $|Q| \leq 1/20$, which forces $|W| \geq (\sqrt{3}/2) w_0$. Near the center, where $|W| \leq \gamma$ and $|Q| \leq 1/400$, the Gaussian multiplier can be inverted. Figure 1 shows the separation between these regions. It lets us compare two exponentials with different constants in their exponents. For now take $0 < \gamma < w_0/8$. We estimate uniformly in s, allowing the prefactor in the interior lower bound to depend on γ.

A spatial average for the upper bound. For a dimensional constant $D \geq 1$, to be fixed below, set

$$P_*(X) = \big((J_{Ds} * |g_T|^2)(X)\big)^{1/d}, \qquad M = 2n + 4.$$

Here and below convolution is in the real spatial variables X. Recall that J_ℓ is the periodization of $C_n \ell^{-n} (1 + |X/\ell|^2)^{-M/2}$, of integral one. Differentiating this positive kernel term by term and then taking the dth root gives

$$|\nabla_X^j P_*| \leq C (Ds)^{-j} P_* \quad (j = 1, 2), \qquad \int P_*^{d+1} \, dX \leq \|g_T\|_{p_{\mathrm{c}}}^{p_{\mathrm{c}}}. \tag{5.4}$$

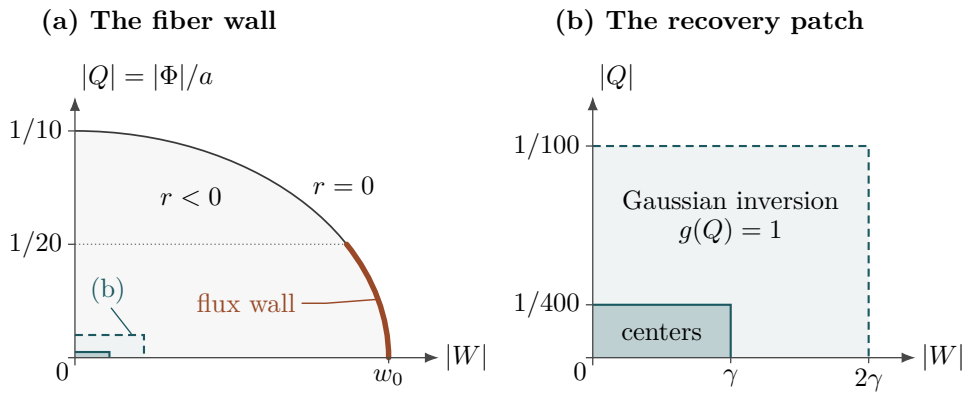


Figure 1: The fixed fiber in modulus coordinates. The wave flux through $r = 0$ is supported on the heavy arc, where $|Q| \le 1/20$ and hence $|W| \ge \sqrt{3}\, w_0/2$. This separation gives Gaussian decay $e^{-cs/h}$ even for $|v| \le w_0/3$. In the enlargement (b), once $K\gamma$ is large, the radius-h balls about the marked centers project into the dashed patch. There $|W - v| \le 4\gamma$ on the enlarged cap $|v| \le 2\gamma$, so Gaussian inversion costs at most $e^{C\gamma^2 s/h}$. The aperture γ is fixed independently of h; its size relative to w_0 is schematic.

The integral bound is Young's inequality with exponent q. The positive kernel gives $P_* > 0$ since $\|g_T\|_2 = 1$.

We next bound the wave current A_{w} and its regularized density μ. By Proposition 2.2, each scalar part of the wave polarization is a multiplier on g_T with symbol a bounded smooth factor times

$$(\kappa/y)^{d/2} \exp(-\Psi), \qquad \Psi = \tfrac{1}{2}\kappa y |W - v|^2, \qquad y = |Y| \asymp s.$$

Multiply the symbol by a fixed smooth annular cutoff equal to one on the data and supported in $|v| < w_0/3$. In the variable $\eta = h\xi$ its derivatives through order M are bounded by $C(hs)^{-d/2}\sigma^m$ for some fixed m. Indeed $h\kappa$ and v have uniformly bounded derivatives on this cutoff, $h\kappa$ is bounded away from zero there, and each derivative of Ψ costs at most $C\sigma$. On $r = 0$ where $g(Q)$ is nonzero, $|W - v|$ has a fixed positive lower bound, so the same estimate gains $e^{-c_0\sigma}$ for some $c_0 > 0$ independent of γ.

If B bounds these derivatives of a compactly supported symbol, applying $(1 - \Delta_\eta)^{n+2}$ in its inverse Fourier transform bounds its absolute kernel by $CBh^{-n}(1 + |X/h|^2)^{-M/2}$. Periodization gives the majorant CBJ_h on the torus, also for a rotated lattice. Thus the squared modulus of the multiplier output is bounded by $CB^2 J_h * |g_T|^2$. There is one more convolution with J_h in A_{w}. We have $J_h * J_h \le CJ_h$: in the Euclidean convolution, either $|Y|$ or $|X - Y|$ is at least $|X|/2$, and the inequality persists after periodization. Moreover $J_h \le (Ds/h)^n J_{Ds}$. Using $(hs)^{-d} = s^{-2d}\sigma^d$ and increasing m when necessary, we obtain

$$\begin{aligned} \|A_{\mathrm{w}}\| &\le Cs^{-2d}\sigma^m P_*^d && (r \le 0), \\ \|A_{\mathrm{w}}\| &\le Cs^{-2d}\sigma^m e^{-c_0\sigma} P_*^d && (r = 0). \end{aligned} \tag{5.5}$$

Here $\|A_{\mathrm{w}}\|$ is the matrix operator norm. Only the first bound is needed for μ. Its extra term is $h^{50n} \det H_{\mathrm{fib}}(r) \le Ch^{50n} a^{-2} s^{-2d}$ by Lemma 4.1. The nearest lattice term in the positive kernel gives $\inf_X J_{Ds}(X) \ge c_D h^{M-n}$, since the torus has fixed diameter and $h \le s \le 3$. Consequently $P_*^d \ge c_D h^{M-n} \|g_T\|_2^2 = c_D h^{M-n}$, which absorbs $h^{50n} a^{-2}$. In particular

$$\mu \le Cs^{-2d}\sigma^m P_*^d \qquad (r \le 0). \tag{5.6}$$

A comparison function that vanishes on the wall. Put $R = H_{\mathrm{fib}}(r)$. Two features of the defining function from Lemma 4.1 will be used together: $R \geq cs^{-2}I$ and $G_{\mathrm{fib}}(r)G_{\mathrm{fib}}(r)^* \leq CR$. It follows that

$$H_{\mathrm{fib}}(rP_*) \geq (1 - C/D)P_*R. \tag{5.7}$$

In fact, the product rule of Lemma 3.1 gives P_*R, $rH_{\mathrm{fib}}(P_*)$, and two terms pairing the gradients of r and P_*. For any quaternionic vector e the two gradient pairings with e have moduli at most $C(e^*Re)^{1/2}$ and $CD^{-1}P_*(e^*Re)^{1/2}$, respectively. This estimates the mixed terms. The term involving $H_{\mathrm{fib}}(P_*)$ has relative cost CD^{-2} by (5.4) and $|r| \leq 1$.

Since $r < 0$, an upper bound on u/r calls for a comparison function below u, with determinant at least $\lambda\mu$. Fix D so the right side of (5.7) is at least $P_*R/2$. The background Hessian $B_{\mathrm{fib}} = \lambda H_{\mathrm{fib}}(|\Phi|^2)$ has rank one and its positive eigenvalue is comparable to $\lambda = h^{-30n}$. Hence for $v_* = C_1\sigma^{m_1}rP_*$,

$$\det\left(B_{\mathrm{fib}} + H_{\mathrm{fib}}(v_*)\right) \geq c\lambda(C_1\sigma^{m_1}P_*s^{-2})^d \geq \lambda\mu$$

by (5.6) with m_1, C_1 large enough. The potentials v_* and u have the same zero boundary data; Dirichlet comparison (Theorem A.1) yields $v_* \leq u \leq 0$. Division by $r < 0$ and passage to the boundary give

$$0 \leq u/r \leq C_1\sigma^{m_1}P_* \qquad (r \leq 0). \tag{5.8}$$

By Lemma 4.3, $\mathcal{F} \leq \int(-u)\mu$ on the fiber. Its imaginary volume element is comparable to $a^2s^{2d}\,dW\,dQ$ by Lemma 4.1. Combining (5.4), (5.6) and (5.8) gives

$$\mathcal{F}(T,\tau) \leq Ca^2\sigma^m\|g_T\|_{p_{\mathrm{c}}}^{p_{\mathrm{c}}}. \tag{5.9}$$

For the flux we need the *full* gradient of r. The tangency in Proposition 2.2 gives $A_{\mathrm{w}}G(|\Phi|^2) = 0$; the remaining part, $G(|W|^2/w_0^2)$, has size $O(s^{-1})$ since $W = P/Y$ and $y \asymp s$. Thus $G(r)^*A_{\mathrm{w}}G(r) \leq Cs^{-2}\|A_{\mathrm{w}}\|$. The Jacobian estimate also gives $\int\delta(r)\,dP\,dZ \leq Ca^2s^{2d}$ by coarea on the fixed ellipsoid in (W,Q). Applying the wall bound in (5.5) and (5.8) to the flux integral in Proposition 4.2, we conclude that

$$\Delta_z\mathcal{F}(T,\tau) \leq C(a/s)^2\sigma^m e^{-c_0\sigma}\|g_T\|_{p_{\mathrm{c}}}^{p_{\mathrm{c}}}. \tag{5.10}$$

Recovering the wave in the interior. Consider a center (X,P,Z) with $|W| \leq \gamma$ and $|Q| \leq 1/400$. The ball of radius h in the $3n$ real spatial coordinates about this center lies in $|W| \leq 2\gamma$, $|Q| \leq 1/100$, provided $K\gamma$ and K are sufficiently large: $W = P/Y$ changes by at most Ch/s and Φ by at most Ch there. The cutoff in A_{w} is one throughout the ball.

The first scalar multiplier S_0 of the time component in (2.15) has symbol $b_0(\kappa/y)^{d/2}e^{-\Psi}$ with $b_0 = (1 + \mathrm{Re}(W)\cdot v/2)/\sqrt{2}$ bounded away from zero. The other scalar component is orthogonal, so $|\mathsf{S}_0 g_T|^2 \leq A^0_{TT}$. Invert S_0 on the frequencies of g_T by multiplying its reciprocal symbol by a smooth annular cutoff equal to one on the data and supported in $|v| \leq 2\gamma$. On its support $\Psi \leq C\gamma^2\sigma$. The derivatives of the reciprocal symbol in $\eta = h\xi$ through order M are bounded by

$$B_{\mathrm{inv}} = C_\gamma(hs)^{d/2}\sigma^m e^{C\gamma^2\sigma}.$$

Here the derivatives of the cutoff enter only the prefactor C_γ; the constant in the exponential is independent of γ. The kernel estimate above yields $|g_T(X)|^2 \leq CB^2_{\mathrm{inv}}(J_h * A^0_{TT})(X)$ for each choice of the imaginary coordinates in the ball. Since $J_h(X' - Y)$ and $J_h(X - Y)$ are comparable for $|X' - X| \leq h$, the density at *every point* in the ball is at least

$$m_X = c_\gamma(hs)^{-d}\sigma^{-m}e^{-C_2\gamma^2\sigma}|g_T(X)|^2. \tag{5.11}$$

For the interior lower bound on $-u$ we need a comparison function *above* u. Let l be Euclidean distance from the center. The quadratic $v_b = \epsilon m_X^{1/d}(l^2 - h^2)$ vanishes on the boundary of the ball and has fiber Hessian $6\epsilon m_X^{1/d} I$ (three real slots per spatial coordinate). For $m_X > 0$, its determinant after adding B_{fib} is at most

$$C((\epsilon m_X^{1/d})^n + \lambda(\epsilon m_X^{1/d})^d) \le \lambda m_X$$

with a fixed small ϵ: the density ceiling (4.7) and (5.11) give $m_X^{1/d} \le \mu^{1/d} \le \lambda$. Now $u \le 0 = v_b$ on the ball boundary and the determinant of $B_{\mathrm{fib}} + H_{\mathrm{fib}}(u)$ is $\lambda\mu \ge \lambda m_X$. The local comparison principle from the proof of Theorem A.1 gives $u \le v_b$ in the ball. At the center the depth is therefore at least $\epsilon h^2 m_X^{1/d}$. Multiplication by $\mu \ge m_X$ gives $(-u)\mu \ge \epsilon h^2 m_X^{1+1/d}$, also when $m_X = 0$. Here $1 + 1/d = q$ and m_X contains $|g_T(X)|^2$, giving the required power $2q = p_{\mathrm{c}}$ of the wave.

The imaginary volume of the center region is comparable to $a^2 s^{2d}\gamma^{2d}$. Integrate the center inequality there and in X, use the lower bound for $\mathcal{F}$ in Lemma 4.3, and recall that $h^2 s^{2d}(hs)^{-dq} = \sigma^{n-2}$. Enlarging the fixed power m and absorbing γ^{2d} into the prefactor gives

$$\mathcal{F}(T,\tau) \ge c_\gamma a^2 \sigma^{-m} e^{-C_3\gamma^2\sigma} \|g_T\|_{p_{\mathrm{c}}}^{p_{\mathrm{c}}}. \tag{5.12}$$

Choose γ so small that $C_3\gamma^2 < c_0/2$, then fix K to meet the ball inclusions and to ensure that $h < 1/(4K)$ lies in the range of (4.7). Dividing the flux bound (5.10) by the interior lower bound (5.12) gives

$$\frac{\Delta_z \mathcal{F}}{\mathcal{F}} \le C_\gamma s^{-2}\sigma^{2m} e^{-(c_0 - C_3\gamma^2)\sigma}.$$

The exponential absorbs the fixed power of σ, proving (5.3). On the two circles in the τ plane the scaled radii $4a/h = 4K$ and $2a/h = 2K$ are fixed. The energy upper bound (5.9) at the first circle gives (5.1) for the initial data; the lower bound (5.12) at the second gives (5.2) for the wave at time T. □

6 Propagation across a thin cylinder

The energy in Proposition 5.1 controls the wave on the circle $|\tau| = 2a$, $1 \le T \le 2$, and is controlled by the data at $|\tau| = 4a$, $T = 0$. To compare these two sets, we first carry the energy out to $|\tau| = 1$, where a fixed neighborhood connects the two time intervals. We then carry it back towards the inner wall. Each passage costs one logarithm, as Figure 2 illustrates. The radial harmonic functions used for comparison are the probabilities that planar Brownian motion leaves an annulus through one circle or the other [57, Theorem 3.18]. We use the maximum principle to retain this comparison in the presence of a potential.

Lemma 6.1. *Let $0 < b < 1/4$ and $D = \log(3/b)$. Suppose a continuous potential $V : [b, 3] \to [0, \infty)$ satisfies*

$$\int_b^3 s\log(s/b)V(s)\,ds \le M, \qquad \sup_{1/2\le s\le 3/2} V(s) \le M. \tag{6.1}$$

Let $E \ge 0$ be smooth on $\mathbb{R}_T \times \{\tau \in \mathbb{R}^2 : b \le |\tau| \le 3\}$, with $\Delta_{T,\tau} E \le V(|\tau|)E$ in the interior. Its circular mean

$$H(T,s) = \frac{1}{2\pi}\int_0^{2\pi} E(T, s(\cos\theta, \sin\theta))\,d\theta$$

satisfies, for $b \le r \le 1$,

$$H(0,r) \;\ge\; c(M)\frac{\log(r/b)}{D^2}\int_1^2 H(T,b)\,dT. \tag{6.2}$$

In particular the factor at $r = 2b$ is a constant times D^{-2}.

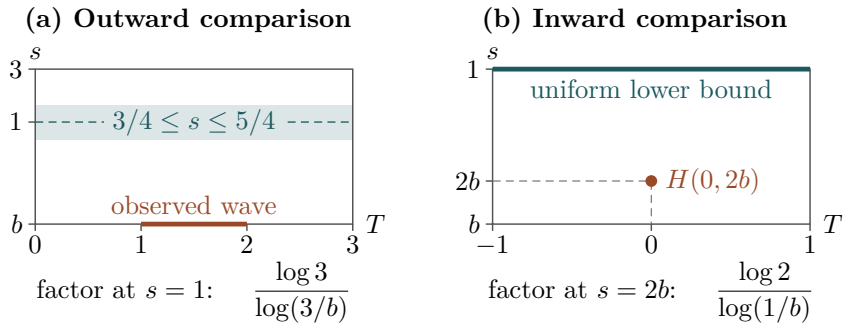


Figure 2: The two comparisons, with radii $s = |\tau|$ drawn schematically. The boundary mass on $1 \le T \le 2$ bounds the mass in the shaded band from below. The mean inequality on a fixed annulus supplies a lower bound on the entire upper wall in (b). The two displayed fractions are values of the normalized radial harmonic functions. The comparison solutions with potential dominate fixed multiples of these values, giving the factor $[\log(3/b)]^{-2}$ at the dot. In the wave estimate, $b = 2Kh$: H controls the wave norm on the marked segment and is controlled by the initial data at the dot.

Proof. Averaging over rotations preserves the differential inequality, so $H(T, |\tau|)$ is again a supersolution of $(-\Delta_{T,\tau} + V)H \ge 0$. We first estimate the radial solutions that will be used for both passages. For $S = 1$ or 3 and a fixed $\kappa > 0$, consider

$$u'' + s^{-1}u' = (V + \kappa^2)u, \qquad b < s < S,$$

with endpoint values $0, 1$ in either order. We claim that u is bounded below by a constant times the function affine in $\log s$ with the same values. Indeed, $x = \log(s/b)$ changes the equation into $u_{xx} = P(x)u$ on $[0, A]$, where $A = \log(S/b)$ and $P(x) = s^2(V(s) + \kappa^2)$. Let ψ solve this equation with $\psi(0) = 0$, $\psi'(0) = 1$. Then $\psi(x) \ge x$, and integrating twice gives

$$\frac{\psi(x)}{x} = 1 + \int_0^x \frac{z(x-z)}{x} P(z) \frac{\psi(z)}{z}\, dz.$$

Since $z(x-z)/x \le \min(z, A-z)$, Gronwall's inequality gives $\psi(A)/A \le \exp(\int_0^A \min(z, A-z)P(z)\, dz)$. The contribution of V to this integral is at most M by (6.1); that of κ^2 is at most $\kappa^2 S^2 \int_0^\infty te^{-2t}\, dt$, using $t = A - z$. Thus $\psi(x)/\psi(A) \ge cx/A$. The integral bound is unchanged under $x \mapsto A - x$, giving the other ordering of the boundary values.

For the outward passage integrate against $\sin(\pi T/3)$ on $[0, 3]$. This cutoff is positive on the observed times and its second derivative is a multiple of itself. Writing $m(s) = \int_0^3 \sin(\pi T/3)H(T, s)\, dT$, two integrations by parts give

$$\int_0^3 \sin(\pi T/3)\, \partial_T^2 H(T, s)\, dT = -(\pi/3)^2 m(s) + (\pi/3)\big(H(0, s) + H(3, s)\big).$$

It follows that $m'' + s^{-1}m' \le (V + \pi^2/9)m$. Since $m(3) \ge 0$, the maximum principle and the radial estimate with $S = 3$ give $m(s) \ge c\, m(b) \log(3/s)/D$. In particular,

$$\int_0^3 \int_{3/4 \le |\tau| \le 5/4} H(T, |\tau|)\, d\tau\, dT \;\ge\; c\, m(b)/D. \tag{6.3}$$

On the fixed region $-2 < T < 4$, $1/2 < |\tau| < 3/2$, we have $\Delta H \le MH$. The compact weak Harnack inequality there gives $H(T, 1) \ge c(M)m(b)/D$ for $|T| \le 1$. To see the comparison directly, add a real variable v: the function $H(T, |\tau|)\cos(\sqrt{M+1}\, v)$ is nonnegative and superharmonic

on a fixed slab about $v = 0$. The mean inequality on the four-dimensional ball of radius 2ρ at $(x, 0)$, integrated over $|v| < \rho$, gives on the three-dimensional ball of radius ρ about $x = (T, \tau)$ the bound $H(x) \geq c\rho^{-3} \int_{B_\rho(x)} H$ for one small radius ρ throughout a neighborhood of the compact connected band $[-1, 3] \times \{3/4 \leq |\tau| \leq 5/4\}$; here $H(x) = H(T, |\tau|)$. Cover the band by finitely many balls of radius $\rho/4$. If two such balls B, B' overlap, then $B' \subset B_\rho(x)$ for every $x \in B$, so $\inf_B H \geq c\rho^{-3} \int_{B'} H$. A finite chain gives the stated lower bound from (6.3).

For the return passage take the radial solution with $S = 1$, $\kappa = \pi/2$, $u(b) = 0$ and $u(1) = 1$. On $|T| < 1$, $b < |\tau| < 1$, the function $c(M)m(b)D^{-1}\cos(\pi T/2)u(|\tau|)$ solves $(-\Delta_{T,\tau} + V)w = 0$. It vanishes on $|T| = 1$ and $|\tau| = b$, and the preceding lower bound puts it below H on $|\tau| = 1$. The maximum principle and the radial estimate for $u(r)$ therefore give

$$H(0, r) \geq \frac{c(M)m(b)}{D} \frac{\log(r/b)}{\log(1/b)} \qquad (b \leq r \leq 1).$$

This proves (6.2), since $\sin(\pi T/3)$ is bounded below on $[1, 2]$. □

Remark 6.2. The dependence on r and b in Lemma 6.1 is sharp for scalar supersolutions, even when $V = 0$. Start with $(1 - \cos T)p(s)$, where $p(s) = \log(3/s)/D$ is the radial harmonic function with boundary values $1, 0$. To cancel the positive part of its Laplacian without changing those boundary values, put $g(s) = s^2(1 + \log(3/s))/(4D)$. The radial harmonic interpolant $L(s) = p(s)g(b) + (1 - p(s))g(3)$ lies above g, since $\Delta_\tau g = p \geq 0$. Thus $E(T, s) = (1 - \cos T)p(s) + L(s) - g(s)$ is bounded and nonnegative, with $\Delta_{T,\tau} E = (\cos T - 1)p \leq 0$. Its trace at $s = b$ is $1 - \cos T$, and since g is increasing we have

$$E(0, r) \leq (1 - p(r))g(3) = \frac{9}{4} \frac{\log(r/b)}{D^2}.$$

A fixed positive r would cost one logarithm. The data control $r = 2b$, where a sharper estimate requires more information about the wave energy.

Proposition 6.3. *On the spatial torus $\mathbb{T}^n = \mathbb{R}^n/(100\mathbb{Z})^n$, let f have Fourier support in $c_0/h \leq |\xi| \leq C_0/h$, $0 < h \leq 1$, and in one cap of the fixed aperture from Proposition 5.1. Then*

$$\left(\int_1^2 \|U(T)f\|_{L^{p_c}(\mathbb{T}^n)}^{p_c} \, dT \right)^{1/p_c} \leq C \, [\log(2/h)]^{2/p_c} \|f\|_{L^{p_c}(\mathbb{T}^n)}. \tag{6.4}$$

The constant is uniform in h and in the direction of the cap.

Proof. For $f \neq 0$, first normalize $\|f\|_{L^2(\mathbb{T}^n)} = 1$. Fix the constant K in Proposition 5.1, put $a = Kh$ and $b = 2a$, and take h small enough that $b < 1/4$. Apply Lemma 6.1 to the energy of that proposition with $V(s) = Cs^{-2}e^{-cs/h}$. Its first bound in (6.1) follows from

$$\int_b^3 s \log(s/b) V(s) \, ds \leq C \int_1^\infty \frac{\log t}{t} e^{-2cKt} \, dt,$$

and V is uniformly bounded when $1/2 \leq s \leq 3/2$. The two energy bounds in Proposition 5.1 are uniform on their respective circles, so its circular mean satisfies

$$H(0, 2b) \leq C_K a^2 \|f\|_{p_c}^{p_c}, \qquad H(T, b) \geq c_K a^2 \|U(T)f\|_{p_c}^{p_c}.$$

Take $r = 2b$ in (6.2), substitute both bounds, cancel a^2, and take p_c-th roots. Since $D = \log(3/(2Kh)) \lesssim \log(2/h)$ and K is fixed, this gives (6.4). Homogeneity removes the normalization after this comparison. For the remaining h the torus frequencies lie in a fixed ball, where the estimate follows by a finite Fourier sum. □

7 Passage to Euclidean space and Sobolev estimates

The aperture in Proposition 6.3 is fixed. A finite partition of the sphere therefore gives the critical estimate on the whole annulus without increasing its logarithmic power. We first remove the spatial period and then account for the cost of summing annuli.

7.1 Removing the period

Proof of Theorem 1.2. Write $h = N^{-1}$ and $P_h = \varphi(hD)$, with φ supported in a fixed compact annulus. Choose a smooth finite partition $\varphi = \sum_\nu \varphi_\nu$ subordinate to the admissible caps. The Euclidean kernel of $\varphi_\nu(hD)$ is $h^{-n}\check{\varphi}_\nu(\cdot/h)$, and periodization does not increase its L^1 norm. Apply Proposition 6.3 to $\varphi_\nu(hD)f$ on the same torus for every ν, and sum. This gives

$$\|U(T)P_h f\|_{L^{p_c}(I\times\mathbb{T}^n)} \lesssim [\log(2/h)]^{2/p_c}\|f\|_{L^{p_c}(\mathbb{T}^n)}, \qquad 0 < h \le 1. \tag{7.1}$$

The Euclidean convolution kernel of $U(T)P_h$ is

$$K_h(T,X) = (2\pi h)^{-n}\int e^{\mathrm{i}(X\cdot\eta - T|\eta|)/h}\varphi(\eta)\,\mathrm{d}\eta.$$

For $T \in I$ and $|X| \ge 3$ the gradient of the phase before division by h has length at least $|X| - 2 \ge |X|/3$. Integration by parts in the gradient direction gains $h/|X|$ each time: the amplitude is supported away from zero and all phase derivatives of order at least two are bounded there. Using at least $M + n$ integrations gives

$$|K_h(T,X)| \le C_M h^M (1+|X|)^{-M}, \qquad T \in I, \quad |X| \ge 3. \tag{7.2}$$

Tile $\mathbb{R}^n$ by unit cubes Q, and write $10Q$ for the concentric cube of side ten. On Q the contribution of $f\mathbf{1}_{(10Q)^c}$ is bounded pointwise by $C_M h^M((1+|\cdot|)^{-M} * |f|)(X)$, since $|X - Y| \ge 9/2$ for $X \in Q$ and $Y \notin 10Q$. This single function bounds the far contributions on every cube. For $M > n$, its $L^{p_c}(I \times \mathbb{R}^n)$ norm is $O(h^M)\|f\|_{p_c}$ by Young's inequality.

Translate the center of Q to the origin and periodize $f_Q = f\mathbf{1}_{10Q}$ on the torus of side 100. Its torus norm is $\|f\|_{L^{p_c}(10Q)}$. The periodic wave kernel is the periodization of K_h. On Q, its nonzero translates see only arguments $X - Y + 100k$ with $Y \in 10Q$ and $k \in \mathbb{Z}^n \setminus \{0\}$. Since $|X_j - Y_j| \le 5.5$, these arguments have length at least $94.5|k|$. Summing (7.2) over k bounds the difference between the two outputs there by $C_M h^M \|f_Q\|_1$. As $10Q$ has fixed volume, (7.1) now gives

$$\|U(T)P_h f_Q\|_{L^{p_c}(I\times Q)} \lesssim [\log(2/h)]^{2/p_c}\|f\|_{L^{p_c}(10Q)}.$$

Take the p_c-th powers and sum over Q. The cubes $10Q$ have bounded overlap. Combining this sum with the far contributions gives the asserted Euclidean estimate, since $\log(2/h) \asymp \log(2+N)$ for $N \ge 2$. □

7.2 Interpolation and summation

Plancherel bounds the norm of $U(T)P_h$ from L^2_X to $L^2_{T,X}$ by $\|\varphi\|_\infty$, since $|I| = 1$. At $p = \infty$ its norm is $O(h^{-d/2})$. To see this, partition the annulus into $O(h^{-d/2})$ caps of aperture $c\sqrt{h}$ with bounded overlap. In coordinates $\eta = (\sqrt{h}\,z, \rho)$ about the axis of a cap, its cutoff is supported in a fixed compact set with ρ bounded below and has uniformly bounded derivatives. Such cutoffs are obtained by dividing rescaled angular bumps by their sum; the overlapping centers have bounded offsets in these coordinates. Removing the linear phase $T\rho/h$ leaves

$$\frac{T}{h}\bigl(\sqrt{\rho^2 + h|z|^2} - \rho\bigr) = \frac{T|z|^2}{\sqrt{\rho^2 + h|z|^2} + \rho}.$$

The expression on the right has uniformly bounded derivatives on the support. Integration by parts thus bounds the L^1 norm of each rescaled kernel by a constant. The affine change of frequencies and the spatial translation from the linear phase preserve that norm. Summing over the caps gives the claimed bound.

Set $s_+(p) = \max\{0, d/2 - n/p\}$ and $\beta(p) = \min\{d(1-2/p), 2/p\}$. Interpolating Theorem 1.2 with the two bounds above gives

$$\|U(T)P_h f\|_{L^p(I\times\mathbb{R}^n)} \lesssim_p h^{-s_+(p)}[\log(2/h)]^{\beta(p)}\|f\|_{L^p(\mathbb{R}^n)}, \qquad 2 \le p \le \infty. \tag{7.3}$$

Indeed, above p_c the power of h^{-1} is $(d/2)(1-p_c/p) = d/2 - n/p$. We use Riesz–Thorin interpolation here and the spatial Littlewood–Paley and Mihlin theorems below; see [32, Chapters 1 and 6].

Proof of Theorem 1.1. The gain of an arbitrary $\varepsilon > 0$ now allows an absolute sum over dyadic annuli. Indeed, set $g = J^{s_+(p)+\varepsilon} f$. At frequency 2^j, $j \ge 1$, (7.3) bounds the output by $C2^{-j\varepsilon}(1+j)^{\beta(p)}\|g\|_p$: a smooth enlargement of the annulus times $2^{j(s_+(p)+\varepsilon)} J^{-s_+(p)-\varepsilon}$ has uniformly integrable kernel. The coefficients are summable. On the remaining low frequencies the cutoff times $e^{-\mathrm{i}T|\xi|}$ obeys uniform Mihlin bounds for $T \in I$. At $p = 2$ use Plancherel without any loss. □

The same summation can be made more precise if one wants an estimate at the exact power $s_+(p)$.

Corollary 7.1 (Logarithmic Sobolev estimate)**.** *Let $2 < p < \infty$, $\delta_p = 1/2 - 1/p$, and $B_*(p) = \beta(p) + \delta_p$. Write $J = (1-\Delta)^{1/2}$, $\ell_1(r) = \log(2+r)$ and $\ell_2(r) = \log(2+\ell_1(r))$. Then*

$$\|U(T)f\|_{L^p(I\times\mathbb{R}^n)} \lesssim_{p,B,C} \|J^{s_+(p)}\ell_1(J)^B\ell_2(J)^C f\|_{L^p(\mathbb{R}^n)} \tag{7.4}$$

whenever $B > B_*(p)$ and $C \in \mathbb{R}$, or $B = B_*(p)$ and $C > \delta_p$. At $p = p_c$, $B_*(p_c) = 1/2 + 1/p_c$ and $\delta_{p_c} = 1/(2n)$. At $p = 2$ no logarithmic factor is needed.

Proof of Corollary 7.1. Let P_j be the projections of a smooth dyadic partition of the high frequencies, $|\xi| \asymp 2^j$, $j \ge 1$, and $\widetilde{P}_j$ an enlarged cutoff equal to one on the support of the symbol of P_j. Write $s = s_+(p)$ and $g = J^s\ell_1(J)^B\ell_2(J)^C f$. On the j-th annulus put

$$A_j = 2^{js}(1+j)^B[\log(2+j)]^C\,\widetilde{P}_j J^{-s}\ell_1(J)^{-B}\ell_2(J)^{-C}.$$

If a_j is the symbol of A_j, then the functions $a_j(2^j\eta)$ have common compact annular support and uniformly bounded derivatives. Bounded overlap of the original annuli therefore gives uniform Mihlin bounds for all signed sums of the A_j. Randomizing these signs and using Khintchine's inequality gives, for $p \ge 2$,

$$\Big(\sum_j \|A_j g\|_p^p\Big)^{1/p} \le \Big\|\Big(\sum_j |A_j g|^2\Big)^{1/2}\Big\|_p \lesssim_p \|g\|_p.$$

The waves $u_j(T) = U(T)P_j f$ retain their spatial frequency supports. The reverse Littlewood–Paley inequality at each T, followed by the triangle inequality in $L^{p/2}(I\times\mathbb{R}^n)$, gives

$$\Big\|\sum_j u_j\Big\|_{L^p_{T,X}} \lesssim_p \Big(\sum_j \|u_j\|_{L^p_{T,X}}^2\Big)^{1/2}.$$

By (7.3), the j-th norm on the right is at most $Cw_j\|A_j g\|_p$, where $w_j = (1+j)^{\beta(p)-B}[\log(2+j)]^{-C}$. With $1/r = \delta_p$, Hölder's inequality bounds this sequence in ℓ^2 by $\|w\|_{\ell^r}(\sum_j \|A_j g\|_p^p)^{1/p}$. The weight

belongs to ℓ^r when $B > B_*(p)$. At $B = B_*(p)$ it belongs to ℓ^r precisely when $\sum_{j\ge 2} j^{-1}(\log j)^{-Cr} < \infty$, that is, $C > \delta_p$.

For the remaining low-frequency cutoff χ_0, the symbol $e^{-\mathrm{i}T|\xi|}\chi_0(\xi)$ obeys the Mihlin derivative bounds uniformly in $T \in I$ away from $\xi = 0$. The same holds after multiplication by the inverse of the Sobolev and logarithmic weights, which are smooth and positive near zero. This proves (7.4), first for finite frequency sums and then by density. At $p = 2$ use conservation of the L^2 norm. □

The condition on B and C records the cost of this frequency summation. It need not be optimal for the wave operator: for $n \ge 4$ and $p > 2 + 4/(n-3)$, Heo, Nazarov and Seeger proved (7.4) with $B = C = 0$ [37, Theorem 1.2].

8 Two obstructions to endpoint smoothing

A spherical wave that focuses at one time determines the necessary Sobolev order. At the critical exponent the wave spends comparable L^{p_c} mass on every dyadic scale between wavelength and unit distance from the focus. We use the radial test of Beltran, Roos, Rutar and Seeger [9, Section 3], keeping the contributions from all these scales.

Proposition 8.1 (Radial focusing). *Let $n \ge 2$, $d = n-1$, and $I = [1,2]$. For every sufficiently large N there is a nonzero radial Schwartz function f_N, with $\operatorname{supp} \widehat{f_N} \subset \{5N/4 < |\xi| < 7N/4\}$, such that*

$$\frac{\|U(t)f_N\|_{L^p(\mathbb{R}^n\times I)}}{\|f_N\|_{L^p(\mathbb{R}^n)}} \gtrsim_{n,p} \begin{cases} 1, & 2 \le p < p_c, \\ (\log N)^{1/p_c}, & p = p_c, \\ N^{d/2-n/p}, & p_c < p < \infty. \end{cases} \tag{8.1}$$

Proof. Fix $t_0 = 3/2$, take a nonzero nonnegative $\psi \in C_c^\infty((5/4, 7/4))$, and set $\widehat{f_N}(\xi) = e^{it_0|\xi|}\psi(|\xi|/N)$. Plancherel gives $\|f_N\|_2 \asymp N^{n/2}$. To bound f_N pointwise, write $\xi = N\sigma\omega$ in its inverse Fourier transform. If $|x| < t_0/2$, then $t_0 + x\cdot\omega \ge t_0/2$ and integration by parts in σ gives $O_m(N^{n-m})$ for every integer $m \ge 0$. If $|x| \ge t_0/2$, the Fourier transform of surface measure on the unit sphere is $O((1+N\sigma|x|)^{-d/2})$. Thus $\|f_N\|_\infty \lesssim N^{(n+1)/2}$, and

$$\|f_N\|_p \le \|f_N\|_2^{2/p}\|f_N\|_\infty^{1-2/p} \lesssim N^{(n+1)/2-1/p} \qquad (2 \le p < \infty). \tag{8.2}$$

The wave at $t = t_0 + u$ has phase $x\cdot\xi - u|\xi|$. Put $r = |x|$ and $F(v) = \int \sigma^{d/2}\psi(\sigma)e^{iv\sigma}\,\mathrm{d}\sigma$. The two stationary points on the sphere give, uniformly for $Nr \ge 1$,

$$U(t_0+u)f_N(x) = c_n N^{(n+1)/2} r^{-d/2}\Big[e^{-i\pi d/4}F\big(N(r-u)\big) + e^{i\pi d/4}F\big(-N(r+u)\big) + O\big((Nr)^{-1}\big)\Big], \tag{8.3}$$

where $c_n > 0$. Indeed the leading terms of the angular integral have phases $\pm N\sigma r$ and amplitudes $(2\pi)^{d/2}(N\sigma r)^{-d/2}e^{\mp i\pi d/4}$; the remainder is $O((N\sigma r)^{-d/2-1})$ [74, Chapter VIII]. Since $F(0) > 0$ and F is Schwartz, there are fixed $\eta > 0$ small and A large such that on the outgoing shell

$$A/N \le u \le 1/4, \qquad |r-u| \le \eta/N,$$

the first term in brackets is bounded below in modulus and dominates the other term and the error. The shell has spatial volume $\asymp N^{-1}u^d$. Consequently (8.2) gives

$$\frac{\|U(t)f_N\|_{L^p(\mathbb{R}^n\times I)}^p}{\|f_N\|_p^p} \gtrsim \int_{A/N}^{1/4} u^{d-dp/2}\,\mathrm{d}u.$$

The exponent of u is -1 precisely for $p = p_c$; evaluating the integral gives all three cases of (8.1). □

On this annulus $J^s = (1-\Delta)^{s/2}$ has operator norm $O_s(N^s)$ on the data, for every real s: after dilation and division by N^s, a slightly enlarged smooth annular cutoff gives a multiplier with uniformly integrable kernel. The same argument applies to $[\log(2+J)]^\beta$, with bound $O_\beta((\log N)^\beta)$. Thus a Sobolev estimate requires $s \geq \max\{0, d/2 - n/p\}$, and at p_c a logarithmic estimate requires $\beta \geq 1/p_c$. The functions f_N are radial, so the same lower power applies to Theorem B.1 with angular derivatives on the input.

For general data, a second example concentrates a fixed amount of L^2 mass on a spacetime set of small measure. Even the weak spacetime norm detects this concentration.

Proposition 8.2 (Weak spacetime obstruction)**.** *For $2 < p \leq p_c$ and every sufficiently large N, there is a nonzero Schwartz function g_N with $\operatorname{supp} \widehat{g}_N \subset \{5N/4 < |\xi| < 7N/4\}$ such that*

$$\frac{\|U(t)g_N\|_{L^{p,\infty}(\mathbb{R}^n \times I)}}{\|g_N\|_{L^p(\mathbb{R}^n)}} \gtrsim_{n,p} (\log N)^{d(1/2-1/p)}. \tag{8.4}$$

The exponent in (8.4) equals $1/p_c$ at p_c and is positive for every $2 < p < p_c$. In particular Sobolev order zero fails throughout $2 < p \leq p_c$. Both lower bounds apply to P_N in Theorem 1.2 whenever $\varphi(\eta) = 1$ for $5/4 \leq |\eta| \leq 7/4$. If the strong critical upper bound held with power $1/p_c$ for all data, interpolation with the L^2 energy bound would give exactly the exponent in (8.4) for $2 < p < p_c$. Appendix C proves Proposition 8.2 by transporting disjoint packets into overlapping tubes.

9 Spherical extension and Kakeya tubes

Write $d = n-1$, $p = p_c = 2n/d$, and $L(R) = \log(2+R)$. We keep the logarithmic power A in the wave estimate as a parameter. To see what happens to it in restriction, we take an incoming spherical wave whose amplitude varies with direction. Near its focus the radial oscillation cancels, leaving the extension of that amplitude from the unit sphere.

For g on S^{n-1} write $Eg(X) = \int_{S^{n-1}} e^{iX\cdot\omega} g(\omega)\, d\sigma(\omega)$. Let $K_\delta f(\omega)$ be the supremum of the averages of $|f|$ over unit tubes of radius δ parallel to ω.

Corollary 9.1 (Spherical extension and Kakeya)**.** *For $n \geq 2$, $R \geq 2$, $z \in \mathbb{R}^n$, and $0 < \delta \leq 1$,*

$$\|Eg\|_{L^{p_c}(B(z,R))} \leq C_n [\log(2+R)]^{2/p_c} \|g\|_{L^{p_c}(S^{n-1})},$$
$$\|K_\delta f\|_{L^n(S^{n-1})} \leq C_n [\log(2/\delta)]^{4/p_c} \|f\|_{L^n(\mathbb{R}^n)}.$$

Every subset of $\mathbb{R}^n$ containing a unit line segment in each direction has Hausdorff dimension n.

A wave estimate with logarithmic power A gives the same power for spherical extension and power $2A$ for the Kakeya maximal operator. The radial example after the proof shows that the latter power must be at least $1 - 1/n$. Thus an unrestricted wave estimate with power $1/p_c$ would give the sharp Kakeya maximal logarithm.

In the plane, Hörmander's estimate gives the sharp logarithmic power $1/4$ for local circle extension: take the output exponent $q = 4 + 1/\log R$ in [38, Corollary 1.3] and apply Hölder's inequality on the observation ball and on the circle. Constant density forces this power by stationary phase. The planar directional Kakeya maximal bound has sharp power $1/2$ [13, Equation (1.5)]; see also Córdoba's theorem for rectangles [19]. Davies proved full Hausdorff dimension in the plane [21], and Wang and Zahl proved it in three dimensions [77, Theorem 1.1]; see also [36]. A separate three-dimensional maximal estimate with $\delta^{-\varepsilon}$ loss and a four-dimensional Hausdorff dimension theorem appear in [63, Theorem 1.1] and [62, Theorem 1.1], respectively. Here the implication from a wave estimate to a fixed logarithmic maximal estimate works in every dimension.

Lemma 9.2. *Suppose (1.2) holds with $L(N)^A$ for some $A \geq 0$, for a smooth radial cutoff $\chi(|\xi|/N)$ with χ supported in $(1/2, 2)$ and equal to one near 1. For $R \geq 2$, $z \in \mathbb{R}^n$, and $g \in L^p(S^{n-1})$, the spherical extension $Eg(X) = \int_{S^{n-1}} e^{iX\cdot\omega} g(\omega)\, d\sigma(\omega)$ satisfies*

$$\|Eg\|_{L^p(B(z,R))} \lesssim L(R)^A \|g\|_{L^p(S^{n-1})}. \tag{9.1}$$

Proof. Choose a nonnegative nonzero $\psi \in C_c^\infty((1,2))$. Modulation in time selects a frequency shell of width of order one:

$$M_N f = \int \psi(T) e^{iNT} U(T) \chi(|D|/N) f \, dT, \qquad \|M_N f\|_p \lesssim L(N)^A \|f\|_p$$

by Hölder in T. For $1/2 \leq |Z| \leq 3$ its convolution kernel is

$$K_N(Z) = c_n N^{d/2} |Z|^{-d/2} e^{iN|Z|} \psi(|Z|) + O(N^{d/2-1}), \qquad c_n \neq 0. \tag{9.2}$$

Indeed, the radial multiplier is $\chi(\rho/N)\widehat{\psi}(\rho - N)$. In polar coordinates the angular integral at $u = \rho|Z|$ is

$$\int_{S^{n-1}} e^{iu\omega_1}\, d\sigma(\omega) = u^{-d/2}(c_+ e^{iu} + c_- e^{-iu}) + O(u^{-d/2-1}), \qquad c_+ c_- \neq 0,$$

by stationary phase at the two poles. After radial integration the remainder is $O(N^{d/2-1})$. In the leading terms one can replace $\chi(\rho/N)\rho^{d/2}$ by $N^{d/2}$ with the same error: the prefactor, extended by zero, has derivative $O(N^{d/2-1})$ and $\int |s\widehat{\psi}(s)|\, ds < \infty$. Fourier inversion gives $\psi(\pm|Z|)$; only the positive argument contributes.

We linearize the distance phase at the focus, as in Hörmander's passage from oscillatory integrals to extension [38, §3, (3.8)–(3.9)]; compare Tao's deduction of local restriction from Bochner–Riesz estimates [75, proof of Theorem 1.1]. Take $f_N(r\omega) = e^{-iNr}\psi(r)g(\omega)$, so that $\|f_N\|_p = C_\psi \|g\|_p$. For $|X| \leq R$ and $N \geq 4R$,

$$N|r\omega - X/N| = Nr - X \cdot \omega + O(R^2/N) \qquad (r \in \operatorname{supp}\psi).$$

Substitute this expansion in (9.2) and integrate in r, ω. The amplitude changes by $O(R/N)$, and the radial phases cancel:

$$N^{-d/2} M_N f_N(X/N) = c' Eg(-X) + O((1+R^2)N^{-1}\|g\|_1), \qquad c' = c_n \int r^{d/2}\psi(r)^2\, dr \neq 0.$$

Choose $N = (4+R)^{3+n/p}$. The error has $L^p(B(0,R))$ norm at most $C\|g\|_p$. On the left the change of variables costs $N^{n/p-d/2} = 1$, and $L(N) \lesssim_n L(R)$. This proves (9.1) at $z = 0$. Replacing $g(\omega)$ by $e^{iz\cdot\omega}g(\omega)$ gives every center. □

9.1 Global extension from the sphere and the truncated cone

Lemma 9.2 gives global estimates at strictly larger output exponents. For this step any fixed power of the logarithm suffices; the tube argument in the next subsection will retain that power. Write E for extension from S^{n-1}. For the truncated cone $\Sigma = \{(\rho\omega, \rho) : 1 \leq \rho \leq 2,\ \omega \in S^{n-1}\}$, let $E_\Sigma G(x,t) = \int_\Sigma e^{i(x\cdot\xi + t\tau)} G(\xi,\tau)\, d\Sigma(\xi,\tau)$, with induced surface measure.

Corollary 9.3 (Global Fourier extension)**.** *Let $1 < a \leq \infty$ and $b < \infty$. If*

$$b > p_c, \qquad b > \frac{n+1}{n-1} a',$$

then

$$\|Eg\|_{L^b(\mathbb{R}^n)} \lesssim_{n,a,b} \|g\|_{L^a(S^{n-1})}, \tag{9.3}$$

$$\|E_\Sigma G\|_{L^b(\mathbb{R}^{n+1})} \lesssim_{n,a,b} \|G\|_{L^a(\Sigma)}. \tag{9.4}$$

These are strong estimates, including when $a = \infty$. For $b = \infty$ both estimates hold for every $1 \le a \le \infty$.

The two strict inequalities give the interior of the conjectured range for these compact surfaces [61, Conjecture 1.1(i)]. The three-dimensional sphere estimate in [64, §§9.1–9.2] gives this range for S^2 and, by randomizing caps, an L^3 Kakeya maximal bound with loss $\delta^{-\varepsilon}$.

Proof. The local spherical estimate in Lemma 9.2, with $A = 2/p_c$ by Theorem 1.2, costs $O_\varepsilon(R^\varepsilon)$ for every $\varepsilon > 0$. By duality it bounds restriction of functions supported in $B(0,R)$ from $L^{p_c'}(\mathbb{R}^n)$ to $L^{p_c'}(S^{n-1})$ with that loss. Since $1 < p_c' < 2$, Tao's epsilon-removal theorem [75, Theorem 1.2] gives global diagonal restriction at every $1 < u < p_c'$. Dualizing gives $E : L^s \to L^s$ for every $p_c < s < \infty$.

If $a \ge b$, take $s = b$ and use the finite measure of the sphere. If $a < b$, put $s = 1 + b/a'$; then $p_c < s < b$, since $p_c = 1 + (n+1)/(n-1)$. Interpolating $E : L^1 \to L^\infty$ with $E : L^s \to L^s$ at $\theta = s/b$ gives (9.3): $\theta/s = 1/b$ and $1 - \theta + \theta/s = 1/a$.

For the cone use Nicola's slicing argument [61, Theorem 1.2(a)(i)]. Write $G_\rho(\omega) = G(\rho\omega, \rho)$. Since $\mathrm{d}\Sigma = \sqrt{2}\,\rho^{n-1}\,\mathrm{d}\rho\,\mathrm{d}\sigma$,

$$E_\Sigma G(x,t) = \sqrt{2}\int_1^2 e^{\mathrm{i}t\rho}\rho^{n-1}E(G_\rho)(\rho x)\,\mathrm{d}\rho.$$

Hausdorff–Young in t, Minkowski's inequality ($b \ge b'$), and spatial dilation give

$$\|E_\Sigma G\|_{L^b_{x,t}} \lesssim \left(\int_1^2 [\rho^{n-1-n/b}\|E(G_\rho)\|_{L^b_x}]^{b'}\,\mathrm{d}\rho\right)^{1/b'} \lesssim \Big\| \|G_\rho\|_{L^a(S^{n-1})} \Big\|_{L^{b'}_\rho([1,2])}.$$

Hölder's inequality gives (9.4) because $b > a'$ implies $a > b'$. For $b = \infty$ the defining integrals are bounded by the L^1 norms on the two compact surfaces. □

9.2 From caps to tubes

Let $T_\delta(\omega, x)$ be the open δ-neighborhood of $x + [-1/2, 1/2]\omega$. Its volume is comparable to δ^d for $0 < \delta \le 1$. Thus $K_\delta f(\omega)$ is the supremum over x of $|T_\delta(\omega,x)|^{-1}\int_{T_\delta(\omega,x)}|f|$. These averages are continuous in the center and direction for $f \in L^n$, by continuity of translations and rotations of the tube indicator in $L^{n/(n-1)}$; in particular the supremum is measurable.

A cap of radius δ produces a wave of roughly constant modulus on a tube of length δ^{-2} and radius δ^{-1}. The restriction argument of Beckner, Carbery, Semmes and Soria [7] and Bourgain [13] puts independent signs on disjoint caps. The expected squared modulus of the sum is then the sum of the squared moduli. This accounts for the factor of two in the logarithmic exponent.

Lemma 9.4. *Assume (9.1) and set $q = p/2 = n/(n-1)$. Let $0 < \delta \le 1/2$, let the ω_j be δ-separated on S^{n-1}, and choose $x_j \in \mathbb{R}^n$ arbitrarily. Then for $T_j = T_{2\delta}(\omega_j, x_j)$ and $a_j \ge 0$,*

$$\Big\|\sum_j a_j \mathbf{1}_{T_j}\Big\|_{L^q(\mathbb{R}^n)} \lesssim L(1/\delta)^{2A}\Big(\delta^d\sum_j a_j^q\Big)^{1/q}. \tag{9.5}$$

Proof. Take the disjoint caps $C_j = \{|\omega - \omega_j| < c\delta\}$ for a small fixed $c > 0$, and put $h_j(\omega) = e^{-\mathrm{i}\delta^{-2}x_j\cdot\omega}\mathbf{1}_{C_j}(\omega)$. For $Y \in T_j$, write $Y - x_j = t\omega_j + v$ with $|t| \le 1/2$, $|v| < 2\delta$. Since $\omega_j \cdot (\omega - \omega_j) = -|\omega - \omega_j|^2/2$, the phase in $Eh_j(\delta^{-2}Y)$ differs from its value at ω_j by at most $c^2/4 + 2c$. Thus $|Eh_j(\delta^{-2}Y)| \gtrsim \sigma(C_j) \asymp \delta^d$ on T_j.

Group the centers by the unit lattice cubes Q, with $J_Q = \{j : x_j \in Q\}$ (use half-open cubes). The tubes in this group lie in the concentric cube $5Q$, so their dilates all lie in a ball of radius $C_n\delta^{-2}$. For independent random signs ε_j set $g_\varepsilon = \sum_{j\in J_Q} \varepsilon_j a_j^{1/2} h_j$. Disjointness of the caps gives $\|g_\varepsilon\|_p^p \lesssim \delta^d \sum_{j\in J_Q} a_j^q$. At each point Jensen's inequality gives $\mathbb{E}|Eg_\varepsilon|^p \ge (\sum_{j\in J_Q} a_j |Eh_j|^2)^q$. Integrate on $\delta^{-2}(5Q)$, change variables back to Y, and apply (9.1) on the enclosing ball:

$$\delta^{dp-2n} \int_{5Q} \Big(\sum_{j\in J_Q} a_j \mathbf{1}_{T_j} \Big)^q \lesssim L(1/\delta)^{Ap}\, \delta^d \sum_{j\in J_Q} a_j^q.$$

The prefactor is one because $dp = 2n$. The cubes $5Q$ have bounded overlap, so the q-th power of the norm on the left of (9.5) is bounded by a constant times the sum of these integrals. Taking the q-th root gives $Ap/q = 2A$. □

9.3 The maximal estimate and Hausdorff dimension

Proof of Corollary 9.1. Theorem 1.2 gives $A = 2/p$ in Lemma 9.2. We keep A in the next steps to record the dependence on the wave estimate. For $0 < \delta \le 1/2$ choose a maximal δ-separated set $\{\omega_j\}$. If $|\omega - \omega_j| \le \delta$, then $T_\delta(\omega, x) \subset T_{2\delta}(\omega_j, x)$. Hence

$$\|K_\delta f\|_{L^n(S^{n-1})} \lesssim \Big(\delta^d \sum_j K_{2\delta} f(\omega_j)^n \Big)^{1/n}.$$

To bound this discrete norm, test by $a_j \ge 0$ with $\delta^d \sum_j a_j^q = 1$. For *any* choice of the centers,

$$\delta^d \sum_j \frac{a_j}{|T_j|} \int_{T_j} |f| \lesssim \int |f| \sum_j a_j \mathbf{1}_{T_j} \lesssim L(1/\delta)^{2A} \|f\|_n$$

by Lemma 9.4 and Hölder. Taking suprema over the centers and then the weights proves the maximal estimate. For $\delta \in [1/2, 1]$ it follows from Hölder on each tube.

Suppose F contains a unit segment in every direction. Its open δ-neighborhood F_δ contains the entire tube around each such segment. Consequently

$$|F_\delta| \gtrsim L(1/\delta)^{-2An} = L(1/\delta)^{-2(n-1)} \qquad (A = 2/p). \tag{9.6}$$

For bounded sets this already gives full lower and upper Minkowski dimension. To obtain Hausdorff dimension we have to allow balls of different sizes in the same cover. In fact the logarithmic rate gives infinite Hausdorff measure for every gauge

$$h_\gamma(r) = r^n [\log(1/r)]^{2An} [\log\log(1/r)]^\gamma, \qquad \gamma > n - 1 \quad (r \text{ small}).$$

Since $h_\gamma(2r) \lesssim h_\gamma(r)$, it suffices to use ball covers. Consider an open ball cover $\{B_j\}$ of F with radii $r_j \le e^{-2^m}$. Group the radii by $J_k = \{j : 2^k \le \log(1/r_j) < 2^{k+1}\}$ as in the planar argument of Keich [44, Claim 3.2]. Put $\delta_k = e^{-2^{k+1}}$, $G_k = \bigcup_{j\in J_k} 2B_j$ and $v_k = \sum_{j\in J_k} r_j^n$. On a segment in direction ω, the length covered by group k is at most $CK_{\delta_k}\mathbf{1}_{G_k}(\omega)$. Indeed, at each covered point the transverse disk of radius δ_k lies in a doubled ball, and disks at different points give disjoint cross

sections of the tube. Summing lengths therefore gives $1 \lesssim \sum_{k\ge m} K_{\delta_k}\mathbf{1}_{G_k}(\omega)$ in every direction. The maximal estimate and Hölder in k yield

$$1 \lesssim \sum_{k\ge m} 2^{2Ak} v_k^{1/n} \lesssim \Big(\sum_j h_\gamma(r_j)\Big)^{1/n} \Big(\sum_{k\ge m} k^{-\gamma/(n-1)}\Big)^{(n-1)/n}.$$

The last factor tends to zero. Thus the Hausdorff measure for this gauge is infinite, and $h_\gamma(r) = o(r^s)$ for every $s < n$ gives $\dim_{\mathrm{H}} F = n$. In the plane the same calculation with the classical L^2 maximal power $1/2$ cited above gives infinite Hausdorff measure for $r^2\log(1/r)[\log\log(1/r)]^\gamma$ whenever $\gamma > 1$. The same covering argument works if a segment in each of a positive measure set of directions meets F in length at least a fixed $\lambda > 0$. □

The radial test $f_\delta(X) = |X|^{-1}\mathbf{1}_{\{2\delta<|X|<1/2\}}$ shows the scale of a possible improvement. Its L^n norm is comparable to $[\log(1/\delta)]^{1/n}$ for small δ, but the average on a tube centered at the origin in any direction is at least $c\int_{4\delta}^{1/4} t^{-1}\,\mathrm{d}t$. Hence the maximal logarithm must have power at least $1 - 1/n = 2/p$. A wave estimate with the sharp power $A = 1/p$ for arbitrary data would attain this bound and give $|F_\delta| \gtrsim L(1/\delta)^{-(n-1)}$. There are compact Kakeya sets satisfying the opposite inequality, by Fernández-Delgado and de la Salle [24, Theorem 1.3].

9.4 Two consequences for other families of segments

For the Nikodym maximal function $N_\delta f(x)$, take the supremum of the same tube averages over all unit segments *through* x. The estimate just proved gives

$$\|N_\delta f\|_{L^n(\mathbb{R}^n)} \lesssim L(1/\delta)^{2A}\|f\|_{L^n(\mathbb{R}^n)}. \tag{9.7}$$

We give the projective argument to keep track of the logarithm; compare Tao [75, Theorem 4.10] and Gao–Liu–Xi [30, §2.2]. First consider only the ρ-neighborhoods of the segments $\{x + u\omega : 1/2 \le u \le 1\}$, and denote the supremum of their averages by $D_\rho f(x)$. By finitely many rotations we can restrict to $|\omega_n| \ge c_n$. Place $x = (x', t_0)$ in a unit cube Q centered at zero and fix t_0. For ρ small, on each such tube $c_n/4 \le |s - t_0| \le 2$ and $|y'| \le C_n$. On either sign of $s - t_0$, the map

$$(y', s) \longmapsto (Y', Y_n) = \big(y'/(s - t_0),\, 1/(s - t_0)\big)$$

and its inverse are uniformly Lipschitz there. It sends the line $y' = x' + (s - t_0)v$ to $Y' = v + Y_n x'$, so the direction $(x', 1)/|(x', 1)|$ is independent of v. The image tube is covered by a bounded number of unit tubes of width $C_n\rho$ in that direction. For each sign, transport f restricted to $5Q \cap \{c_n/4 \le \pm(s - t_0) \le 2\}$ by this map and extend by zero; this is a single input for all x' in the slice. Its L^n norm is at most $C\|f\|_{L^n(5Q)}$ by the Jacobian bounds. The map from x' to directions has area Jacobian bounded below. Applying the Kakeya estimate in each slice, integrating in t_0, then translating and summing over cubes, gives $\|D_\rho f\|_n \lesssim L(1/\rho)^{2A}\|f\|_n$. For ρ bounded below the same estimate follows by averaging over a fixed ball about x.

Return to an arbitrary segment through x. Separate the part within $O(\delta)$ of x, then divide each of the two rays into distances $s/2 \le u \le s$ for dyadic $\delta \le s \le 1$. Relative to the full tube volume, the innermost ball has weight $O(\delta)$, and a piece at distance s has weight $O(s)$. By dilation the latter's maximal average has the norm of $D_{\delta/s}$. Thus the operator norm is bounded by $C\delta + C\sum_s sL(s/\delta)^{2A} \lesssim L(1/\delta)^{2A}$, proving (9.7). The transverse disk proof above now applies to any Borel set meeting a suitable unit segment through each point of a positive measure set in length at least $\lambda > 0$. Such a set has Hausdorff dimension n. The Nikodym maximal estimate and this dimension conclusion hold locally on a manifold of constant sectional curvature for geodesic

segments of fixed short length in a compact subset of a straightening chart: there lengths, tube widths and volumes are comparable before and after straightening [30, Proposition 2.3].

For a different application, fix $n \geq 3$ and a real $d \times d$ matrix B. Consider compact sets containing for every $y \in [-1,1]^d$ one curve

$$\{(w(y) + (tI_d + t^2 B)y, t) : |t| \leq \tau_B\},$$

where $\tau_B > 0$ is fixed sufficiently small and $w(y)$ belongs to the fixed sufficiently large compact ball of [59, Equation (1.16)]. Nadjimzadah's lifting theorem [59, Theorem 1.3] now applies, since Corollary 9.1 gives full Kakeya dimension in $n + \operatorname{rank}(B^2)$ dimensions. The minimum Hausdorff dimension in this curve family is attained and is

$$n - \operatorname{rank} B + \operatorname{rank}(B^2) = n - \dim(\operatorname{im} B \cap \ker B);$$

the last equality is rank–nullity for B on its image.

10 Bochner–Riesz means

The Bochner–Riesz mean $B_r^\alpha f$ has Fourier multiplier $(1 - |\xi|^2/r^2)_+^\alpha$; write $B_*^\alpha f = \sup_{r>0} |B_r^\alpha f|$. Local smoothing controls these means because a smooth cutoff to a shell of relative width δ can be written as an average of waves for times of order δ^{-1}. Sogge observed the implication for a single radius [72, Equations (4.5)–(4.6)]; Beltran, Hickman and Sogge give both the fixed-radius and maximal reductions [8, Propositions 39 and 40]. In the range $p \geq 2$, almost everywhere convergence at the strict orders below was proved by Carbery, Rubio de Francia and Vega [17].

Theorem 10.1 (Bochner–Riesz means). *Let $n \geq 2$, $1 \leq p \leq \infty$, and put $\delta(p) = \max\{0, n|1/p - 1/2| - 1/2\}$. If $\alpha > \delta(p)$, then*

$$\sup_{r>0} \|B_r^\alpha f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p,\alpha} \|f\|_{L^p(\mathbb{R}^n)}. \tag{10.1}$$

In this range $B_r^\alpha f \to f$ in L^p as $r \to \infty$ if $p < \infty$. If $2 \leq p < \infty$ and $\alpha > s_+(p)$, then

$$\|B_*^\alpha f\|_{L^p(\mathbb{R}^n)} \leq C_{n,p,\alpha} \|f\|_{L^p(\mathbb{R}^n)}, \tag{10.2}$$

and $B_r^\alpha f \to f$ almost everywhere as $r \to \infty$.

Here and in the square function estimate of Section 11, the strict inequality on the order of the means leaves room for a small Sobolev loss. We rescale Theorem 1.1 to a unit frequency annulus and a long time interval. For $2 \leq p < \infty$, $\sigma > s_+(p) = \max\{0, (n-1)/2 - n/p\}$, and $\widehat{g}$ supported in a fixed annulus $c \leq |\xi| \leq C$,

$$\|U(t)g\|_{L^p(\mathbb{R}^n \times \{R \leq |t| \leq 2R\})} \lesssim_{c,C,n,p,\sigma} R^{1/p+\sigma} \|g\|_p, \qquad R \geq 1. \tag{10.3}$$

Indeed $g(R\cdot)$ has frequencies of order R and $U(t)[g(R\cdot)](x) = (U(Rt)g)(Rx)$. Changing variables in space and time contributes $R^{(n+1)/p}$, while the input norm contributes $R^{-n/p}$ and the Sobolev multiplier on that annulus costs $O(R^\sigma)$. Conjugation gives the estimate for negative times. For $|t| \leq 1$ the unit annular wave multiplier has a uniformly integrable kernel.

Lemma 10.2 (Maximal estimate for a thin shell). *For $0 < \delta \leq 1/8$ and $\psi \in C_c^\infty((1/2, 2))$, set*

$$A_{\delta,r} f = \psi((1 - |D|/r)/\delta) f.$$

For $2 \leq p < \infty$ and $\sigma > s_+(p)$,

$$\Big\| \sup_{r>0} |A_{\delta,r} f| \Big\|_p \lesssim \delta^{-\sigma} \|f\|_p.$$

The constant is uniform for ψ in bounded subsets of $C_c^\infty((1/2, 2))$.

Proof. For $1 \le r \le 2$ the shell lies in $1/2 \le |\xi| \le 2$. Choose a smooth annular symbol χ equal to one there, and write $Q_k = \chi(2^{-k}D)$. For $g = Q_0 f$, Fourier inversion gives

$$A_{\delta,r} f(x) = \frac{\delta r}{2\pi} \int_{\mathbb{R}} e^{itr} \widehat{\psi}(\delta r t) U(t) g(x) \, \mathrm{d}t.$$

Taking the supremum over $r \in [1,2]$ bounds this by $C_M \delta \int (1+\delta|t|)^{-M} |U(t)g(x)| \, \mathrm{d}t$. On a dyadic interval $R \le |t| \le 2R$, Hölder's inequality in time and (10.3) bound the L^p_x norm of this contribution by

$$C_M \delta (1+\delta R)^{-M} R^{1-1/p} R^{1/p+\sigma} \|g\|_p.$$

For $M > 1+\sigma$ its sum over dyadic $R \ge 1$ is $O(\delta^{-\sigma}\|g\|_p)$, by splitting at $R = \delta^{-1}$. The interval $|t| \le 1$ contributes $O(\delta \|g\|_p)$.

For $r = 2^k s$, $1 \le s \le 2$, set $g_k(y) = (Q_k f)(2^{-k}y)$. Then $A_{\delta,r} f(2^{-k}y) = A_{\delta,s} g_k(y)$. The factor $2^{-kn/p}$ in the output norm cancels the factor $2^{kn/p}$ in $\|g_k\|_p$. Thus the L^p norm of the supremum in this octave is at most $C\delta^{-\sigma}\|Q_k f\|_p$. The supremum over all octaves is bounded by their pointwise ℓ^p norm. Since $p \ge 2$, Littlewood–Paley theory gives

$$\left(\sum_{k \in \mathbb{Z}} \|Q_k f\|_p^p \right)^{1/p} \le \left\| \left(\sum_{k \in \mathbb{Z}} |Q_k f|^2 \right)^{1/2} \right\|_p \lesssim \|f\|_p. \qquad \square$$

Proof of Theorem 10.1. Take $2 \le p < \infty$ and choose $s_+(p) < \sigma < \alpha$. To decompose $(1-\rho^2)^\alpha_+$, choose $\eta \in C_c^\infty((1/2,2))$ with $\sum_{j \in \mathbb{Z}} \eta(2^j v) = 1$ for $v > 0$, and apply the partition to $v = 1-\rho$ on $0 \le \rho < 1$. For $\delta = 2^{-j} \le 1/8$ the corresponding term is

$$\delta^\alpha \psi_\delta((1-\rho)/\delta), \qquad \psi_\delta(u) = \eta(u) u^\alpha (2-\delta u)^\alpha.$$

These profiles have uniform smooth seminorms. The remainder is a smooth radial multiplier of compact support: it vanishes near $\rho = 1$ and equals $(1-\rho^2)^\alpha$ near zero. The supremum of its dilations is bounded pointwise by a constant times the Hardy–Littlewood maximal function, since its kernel has a radial decreasing integrable majorant. Lemma 10.2 bounds the other terms in (10.2) by the convergent sum $C \sum_{j \ge 3} 2^{-j(\alpha-\sigma)} \|f\|_p$.

For $2 \le p < \infty$, $\delta(p) = s_+(p)$, so the maximal estimate implies (10.1). Each fixed B_r^α is self-adjoint on Schwartz functions, and $\delta(p) = \delta(p')$; duality proves (10.1) for $1 < p < 2$. At $p = 1, \infty$ the hypothesis is $\alpha > (n-1)/2$. The unit-radius kernel is a constant multiple of $|x|^{-n/2-\alpha} J_{n/2+\alpha}(|x|)$, by radial Fourier inversion. It is bounded at zero and decays as $O(|x|^{-(n+1)/2-\alpha})$ at infinity. Its dilates $r^n K(rx)$ therefore have a common L^1 norm.

For Schwartz functions of compact Fourier support the multipliers converge to one in all derivatives on the support as $r \to \infty$; hence the means converge in the Schwartz topology. Density and (10.1) give norm convergence for $p < \infty$. To see pointwise convergence in the maximal range, choose such functions f_j with $\sum_j \|f - f_j\|_p < \infty$. By (10.2), $\sum_j B_*^\alpha (f_{j+1} - f_j)$ is finite almost everywhere. At every such point $r \mapsto B_r^\alpha f_j(x)$ converges uniformly on $r > 0$ to a continuous representative of the means, agreeing with the L^p extension for each fixed r. Its supremum is measurable by continuity, and Fatou's lemma extends (10.2) to f. Since $f_j(x) \to f(x)$ almost everywhere, uniform approximation in r gives the asserted almost everywhere limit. $\square$

11 Stein's square function

Stein introduced a square function to study the convergence of Bochner–Riesz means [73, §18, (18.1)]. For the Euclidean means B_r^α, with multiplier $(1-|\xi|^2/r^2)^\alpha_+$, it measures the change with

radius:

$$\mathcal{G}_\alpha f(x) = \left(\int_0^\infty |r \partial_r B_r^\alpha f(x)|^2 \frac{\mathrm{d}r}{r} \right)^{1/2}.$$

Carbery proved the sharp bounds in the plane [16, pp. 409–416]. In dimensions $n \geq 3$, Lee, Rogers and Seeger proved them for $p \geq 2 + 4/n$ [50, Theorem 1.1]. Gan, Jing and Wu reached $p \geq 13/4$ in dimension three [26, Theorem 1.1], and Gan, Oh and Wu further enlarged the range in dimensions $n \geq 4$ [27, Theorem 1.1]. Here the wave estimate (10.3) gives the full range $2 \leq p < \infty$.

Theorem 11.1. *Let $n \geq 2$ and $2 \leq p < \infty$. If*

$$\alpha > \max\left\{ \tfrac{1}{2},\, n\left(\tfrac{1}{2} - \tfrac{1}{p}\right) \right\},$$

then $\|\mathcal{G}_\alpha f\|_p \leq C_{n,p,\alpha} \|f\|_p$ for $f \in \mathcal{S}(\mathbb{R}^n)$, and $\mathcal{G}_\alpha$ extends to $L^p(\mathbb{R}^n)$.

The threshold can be read off from one shell. At distance δ from $|\xi| = s$, differentiating the mean gives a factor $\delta^{\alpha-1}$. The square norm in s gives back $\delta^{1/2}$, while the wave estimate at times of order δ^{-1} costs $\delta^{-s_+(p)-\varepsilon}$. The resulting powers sum if $\alpha > 1/2 + s_+(p)$. We first make this calculation on $1 \leq s \leq 2$. We then combine the dyadic intervals of radii inside the square norm, before taking the spatial L^p norm.

Lemma 11.2 (One interval of radii)**.** *Let $2 < p < \infty$, $\varepsilon > 0$, $\sigma = s_+(p) + \varepsilon$, and $H = L^2([1,2], \mathrm{d}s)$. Fix $A \in C_c^\infty(\mathbb{R} \times (0,\infty))$ and $\varphi \in C_c^\infty(\mathbb{R})$. For $0 < \delta \leq 1$, let $T_\delta f(x,s)$ be the spatial Fourier multiplier*

$$A(s, |\xi|)\, \varphi((s - |\xi|)/\delta)$$

applied to f. Then

$$\|T_\delta f\|_{L_x^p(H)} \lesssim \delta^{1/2-\sigma} \|f\|_p, \qquad \|T_\delta f\|_{L_x^2(H)} \lesssim \delta^{1/2} \|f\|_2,$$

with constants independent of δ.

Proof. Temporarily omit A, allow $s \in \mathbb{R}$, and denote the resulting operator by V_δ. For g supported in a fixed compact annulus in frequency, Plancherel in s gives, at each x,

$$\|V_\delta g(x,\cdot)\|_2 = (2\pi)^{-1/2} \delta \|\widehat{\varphi}(\delta t) U(t) g(x)\|_{L_t^2}.$$

On $R \leq |t| \leq 2R$, $R \geq 1$, Hölder in t and the rescaled wave estimate (10.3) give

$$\|U(t)g\|_{L_x^p L_t^2} \lesssim R^{1/2-1/p} R^{1/p+\sigma} \|g\|_p.$$

On $|t| \leq 1$ the annular wave kernels have uniformly bounded L^1 norms. For $b > 1/2 + \sigma$, summing over $R = 1, 2, 4, \ldots$ now gives

$$\|V_\delta g\|_{L_x^p L_s^2(\mathbb{R})} \lesssim \delta \left(1 + \sum_R R^{1/2+\sigma} (1 + \delta R)^{-b} \right) \|g\|_p \lesssim \delta^{1/2-\sigma} \|g\|_p.$$

To restore A, Fourier invert it in s. The multipliers $\widehat{A}_1(\eta, |D|)$, where the hat denotes transformation in the first argument, are supported in a fixed annulus and have L^p operator norms $O_b((1+|\eta|)^{-b})$ for every $b > 0$, by smoothness and compact support. Integrating $e^{\mathrm{i}s\eta} V_\delta(\widehat{A}_1(\eta, |D|) f)$ in η with measure $\mathrm{d}\eta/(2\pi)$ proves the first bound. For the second, spatial Plancherel suffices: at each frequency the multiplier is bounded and is supported on an s-interval of length $O(\delta)$. □

To combine the intervals of radii, we use Seeger's method for passing from bounds on individual frequency annuli to a multiplier estimate [67, Theorem 1]. Jones, Seeger and Wright stated a square function version for dilates of annular multipliers [40, p. 6737]; Guo, Roos and Yung proved it [34, Proposition 4.2 and §7]. Here we allow a different input at each scale and assume decay of the Hilbert-valued kernel and its first derivative. On each test cube only $O(1+\log(1/\delta))$ scales need the L^p bound. Hölder over those scales gives the logarithmic power $1/2-1/p$, which the strict inequality in Theorem 11.1 absorbs.

Lemma 11.3 (Combining dilates)**.** *Let H be a Hilbert space, $2<p<\infty$, $0<\delta\le 1$, and $N=n+3$. Suppose $K:\mathbb{R}^n\to H$ is continuously differentiable and*

$$\|K(x)\|_H+\|\nabla K(x)\|_{H^n}\le C_*\delta^{-N}(1+|x|)^{-N}.$$

Write $S^{(k)}f=2^{nk}K(2^k\cdot)*f$, and let B_q bound the norm of $K*$ from L^q to $L^q(H)$, $q=2,p$. For every finite sequence of Schwartz functions f_k,

$$\left\|\left(\sum_k\|S^{(k)}f_k\|_H^2\right)^{1/2}\right\|_p\lesssim_{n,p}\left(B_2+C_*\delta+[1+\log(1/\delta)]^{1/2-1/p}B_p\right)\left\|\left(\sum_k|f_k|^2\right)^{1/2}\right\|_p.$$

Proof. Put $b_k=\|S^{(k)}f_k\|_H$, $h=(\sum_k b_k^2)^{1/2}$, and $W=(\sum_k|f_k|^2)^{1/2}$. Let M denote the Hardy–Littlewood maximal operator over cubes and $L=\lceil(N+2)\log_2(2/\delta)\rceil$. On a dyadic cube Q of side 2^{-j} and center y, the terms with $k\le j-L$ are nearly constant. We compare h with $c_Q=(\sum_{k\le j-L}b_k(y)^2)^{1/2}$. For $x,z\in Q$ and $k\le j-L$, the derivative bound gives

$$\|S^{(k)}f_k(x)-S^{(k)}f_k(y)\|_H\lesssim C_*\delta^{-N}2^{k-j}MW(z).$$

Indeed $2^k|x-z|\lesssim 1$, and convolution with $2^{nk}(1+2^k|\cdot|)^{-N}$ is bounded by a constant times M. Summing the squares and taking their square root costs at most $CC_*\delta MW(z)$.

For $k\ge j+L$, split f_k on $5Q$ and its complement. The local terms satisfy

$$\frac{1}{|Q|}\int_Q\sum_{k\ge j+L}\|S^{(k)}(f_k\mathbf{1}_{5Q})(x)\|_H^2\,\mathrm{d}x\lesssim B_2^2M(W^2)(z),\qquad z\in Q,$$

because dilation preserves the L^2 operator norm. For the terms outside $5Q$, summing the kernel bound on successive doubled cubes gives, for $x,z\in Q$,

$$\|S^{(k)}(f_k\mathbf{1}_{(5Q)^c})(x)\|_H\lesssim C_*\delta^{-N}2^{-(N-n)(k-j)}MW(z).$$

Their square sum again has square root at most $CC_*\delta MW(z)$.

There remain fewer than $2L$ indices. Hölder in this finite set bounds $\big(|Q|^{-1}\int_Q\sum_{|k-j|<L}b_k^2\big)^{1/2}$ by $(2L)^{1/2-1/p}F(z)$, where $F=(\sum_k[M(b_k^2)]^{p/2})^{1/p}$. Since $p/2>1$, the scalar maximal theorem and the scale invariance of B_p give

$$\|F\|_p^p\lesssim\sum_k\|b_k\|_p^p\le B_p^p\sum_k\|f_k\|_p^p\le B_p^p\|W\|_p^p.$$

By the triangle inequality for the square norm and Cauchy–Schwarz on Q, the mean of $|h-c_Q|$ is thus at most a constant times $C_*\delta MW(z)+B_2M(W^2)(z)^{1/2}+L^{1/2-1/p}F(z)$. Replacing c_Q by the mean h_Q costs at most a factor of two. The dyadic sharp function inequality [74, Chapter IV, §3.6, Corollary 1] and the maximal theorem on L^p and $L^{p/2}$ now prove the assertion. The hypothesis of that inequality at a smaller exponent holds: $\|h\|_2^2=\sum_k\|S^{(k)}f_k\|_{L^2(H)}^2\le B_2^2\sum_k\|f_k\|_2^2<\infty$, and $2<p$. □

Proof of Theorem 11.1. For $p = 2$, spatial Plancherel and $u = |\xi|^2/r^2$ give the identity

$$\|\mathcal{G}_\alpha f\|_2^2 = 2\alpha^2 \int_0^1 u(1-u)^{2\alpha-2}\,\mathrm{d}u\,\|f\|_2^2 = \frac{\alpha}{2\alpha-1}\|f\|_2^2.$$

Suppose $p > 2$. Choose $\varepsilon > 0$ so that $\sigma = s_+(p) + \varepsilon < \alpha - 1/2$, possible since $1/2 + s_+(p) = \max\{1/2, n(1/2 - 1/p)\}$. Write $r = 2^k s$, $1 \le s \le 2$, and $\rho = 2^{-k}|\xi|$. On this interval $\mathrm{d}r/r = \mathrm{d}s/s$ is comparable to $\mathrm{d}s$. The differentiated multiplier equals

$$m(s,\rho) = 2\alpha\rho^2 s^{-2\alpha}(s+\rho)^{\alpha-1}(s-\rho)^{\alpha-1}\mathbf{1}_{\{\rho<s\}},$$

where the last power is taken only for $\rho < s$. Near the boundary $\rho = s$, split $s - \rho > 0$ into dyadic intervals of size $\delta = 2^{-j}$, $j \ge 0$. Each piece is $\delta^{\alpha-1}A(s,\rho)\varphi((s-\rho)/\delta)$: the factors preceding $(s-\rho)^{\alpha-1}$ times a smooth boundary cutoff give a fixed amplitude A supported away from $\rho = 0$; the power on a unit shell is absorbed in $\varphi \in C_c^\infty((1/2, 2))$.

For fixed δ, Lemma 11.2 bounds the one-interval norms of T_δ. Its kernel and first spatial derivatives satisfy the hypothesis of Lemma 11.3 with C_* independent of δ: N integrations by parts in frequency cost at most δ^{-N} on a fixed compact annulus, uniformly in s. Write $T_\delta^{(k)}$ for the spatial dilation obtained by replacing $|\xi|$ by $2^{-k}|\xi|$ in the symbol of T_δ, with $s \in [1, 2]$ unchanged. Choose smooth annular projections P_k, dilates of one fixed multiplier, which equal one on this spatial frequency support. Lemma 11.3 applied to $P_k f$ and the Littlewood–Paley inequality bound this shell, on all dyadic radius intervals together, by

$$C\delta^{\alpha-1}\left(\delta^{1/2} + \delta + [1 + \log(1/\delta)]^{1/2-1/p}\delta^{1/2-\sigma}\right)\|f\|_p.$$

The sum over j converges because $\alpha - 1/2 - \sigma > 0$.

The multiplier left away from the boundary is smooth, supported in $\rho \le 2$, and vanishes to order two at zero. Split it into annuli $\rho \asymp 2^\ell$, $\ell \le 2$. After extracting $2^{2\ell}$ and rescaling to the unit annulus, the multipliers have uniformly bounded derivatives. Their kernels satisfy Lemma 11.3 at $\delta = 1$ with uniform constants (also for B_2, B_p, by the integrable kernel bound). For each ℓ their dilates now have frequency scale $2^{k+\ell}$; the same Littlewood–Paley estimate applies, and $\sum_{\ell\le 2} 2^{2\ell} < \infty$.

These estimates apply first to finite sets of k and finite shell sums. For Schwartz f the multipliers converge in the Fourier integral for each fixed r, by the local integrability of the boundary power. Fatou's lemma and then monotone convergence give the full estimate. The linear map $f \mapsto (r\partial_r B_r^\alpha f)_{r>0}$ therefore extends by density to $L_x^p(L^2(\,\mathrm{d}r/r))$, giving the asserted extension of its pointwise norm. □

12 The paraboloid and Schrödinger evolution

We return to the spherical extension estimate of Lemma 9.2, with $n = d + 1$. A small cap on S^d approaches the paraboloid under parabolic rescaling [38, §3, (3.10)–(3.12)]. The Fresnel formula then writes a Schrödinger wave in d spatial dimensions as extension from that paraboloid. At $p_c = 2(d+1)/d$ both changes of variables preserve the logarithmic power. We use the relation between paraboloid restriction and Schrödinger evolution developed by Rogers [66] and Lee, Rogers and Seeger [51].

Write B_d for the unit ball in $\mathbb{R}^d$ and define extension from the compact paraboloid patch by

$$Eg(x,t) = \int_{B_d} e^{i(x\cdot v + t|v|^2)} g(v)\,\mathrm{d}v.$$

Corollary 12.1 (Paraboloid extension and Schrödinger local smoothing)**.** *Let $d \geq 1$ and $p_c = 2(d+1)/d$. If $1 < a \leq \infty$, $p_c < b < \infty$ and $b > (d+2)a'/d$, then*

$$\|Eg\|_{L^b(\mathbb{R}^{d+1})} \lesssim_{d,a,b} \|g\|_{L^a(B_d)}. \tag{12.1}$$

Let $I \subset \mathbb{R}$ be a compact interval. For $N \geq 2$ and Schwartz f with $\operatorname{supp} \widehat{f} \subset B(0,N) \subset \mathbb{R}^d$,

$$\|e^{it\Delta} f\|_{L^{p_c}(\mathbb{R}^d \times I)} \lesssim_{d,I} [\log(2+N)]^{2/p_c} \|f\|_{p_c}. \tag{12.2}$$

For $p_c < p < \infty$ and $p \leq r \leq \infty$, arbitrary Schwartz data satisfy

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d; L^r(I))} \lesssim_{d,p,r,I} \|f\|_{B^s_{p,p}(\mathbb{R}^d)} \lesssim_{d,p,s} \|(1-\Delta)^{s/2} f\|_p, \qquad s = d(1-2/p) - 2/r. \tag{12.3}$$

Here $B^s_{p,p}$ is the inhomogeneous Besov space. Taking $r = p$ gives the exact spacetime Sobolev order $d - 2(d+1)/p$; taking $r = \infty$ gives the corresponding maximal estimate.

Proof. We first transfer Lemma 9.2 to E, writing A for the logarithmic power in that lemma. By Theorem 1.2 we may take $A = 2/p_c$. Parametrize a spherical cap by $(\delta v, \sqrt{1-\delta^2|v|^2})$ and evaluate its extension at $(x/\delta, -2t/\delta^2)$. After removing a scalar phase, the new graph is $q_\delta(v) = 2(1-\sqrt{1-\delta^2|v|^2})/\delta^2$. Choose the cap density to be $g(v)\sqrt{1-\delta^2|v|^2}$, so surface measure times density is $\delta^d g(v)\,\mathrm{d}v$. The image of $B(0,R)$ in physical space lies in $B(0, CR/\delta^2)$. Thus the extension operator for q_δ has local $L^{p_c} \to L^{p_c}$ norm at most

$$C\,\delta^{-d+(d+2)/p_c + d/p_c} [\log(2 + R/\delta^2)]^A.$$

The power of δ is zero. Since $q_\delta(v) = |v|^2 + O(\delta^2)$, the error in replacing its extension by Eg on $B(0,R)$ is at most $CR\delta^2\|g\|_1$ pointwise. Taking $\delta = (2R)^{-(d+2)}$ makes its L^{p_c} norm $O(\|g\|_{p_c})$ and keeps $\log(2 + R/\delta^2) \lesssim_d \log(2+R)$. Modulation of g translates the observation ball, so

$$\|Eg\|_{L^{p_c}(B(z,R))} \lesssim_{d,A} [\log(2+R)]^A \|g\|_{p_c} \qquad (z \in \mathbb{R}^{d+1},\ R \geq 1). \tag{12.4}$$

To obtain (12.1), first note that the loss in (12.4) is $O_\varepsilon(R^\varepsilon)$ for every $\varepsilon > 0$, also on a slightly larger parameter ball by dilation. A smooth density supported on the larger patch and equal to $\mathrm{d}v$ on B_d has Fourier decay $O((1+|(x,t)|)^{-d/2})$ by stationary phase. Apply Tao's epsilon-removal theorem [75, Theorem 1.2] in the compact hypersurface form of [46, Theorem 5.3], with both restriction exponents equal to $p_c' \in (1,2)$. Duality and zero extension from B_d give $E : L^{p_c}(B_d) \to L^s(\mathbb{R}^{d+1})$ for every $s > p_c$. Since B_d has finite measure, $E : L^s \to L^s$ as well. Interpolation with $E : L^1 \to L^\infty$ now gives (12.1) as in the proof of Corollary 9.3, with $n = d+1$.

For the critical estimate write $p = p_c$, $S(t) = e^{it\Delta}$ and $I_0 = [1/2, 1]$. We give the finite-scale comparison directly from the Fresnel kernel; compare [51, Lemma 2.2]. Let $\chi \in C_c^\infty(B(0,2))$ equal one on $B(0,1)$, so $f = P_N f$ for $P_N = \chi(D/N)$. Tile $\mathbb{R}^d$ by cubes Q of side N. For $t \in I_0$ and $|z| \geq 8N$, integration by parts bounds the kernel of $S(t)P_N$ by $C_M N^d (1+N|z|)^{-M}$: in the frequency variable $\eta = \xi/N$ the phase gradient $Nz - 2tN^2\eta$ has length at least $N|z|/2$ on the support of χ. Thus on each Q the contribution from outside the concentric cube $30Q$ is bounded by the convolution of this majorant with $|f|$. The same convolution works on every cube and has L^p norm $O(\|f\|_p)$ by Young's inequality for $M > d$.

Translate Q to be centered at zero and set $h = f\mathbf{1}_{30Q}$. This truncation may introduce higher frequencies, so the near contribution is $S(t)P_N h = P_N S(t) h$. The equality holds on L^2; at x the last expression integrates $S(t)h(x - w/N)$ against $\check{\chi}(w)\,\mathrm{d}w$. Choose $\lambda = 16\sqrt{d}\,N$, so $g(v) = h(\lambda v)$ is supported in the unit ball. Since $h \in L^1 \cap L^2$, its Fresnel integral converges absolutely. Substitution of $y = \lambda v$ gives

$$|S(t)h(z)| = (4\pi t)^{-d/2} \lambda^d \left| Eg\left(-\frac{\lambda z}{2t}, \frac{\lambda^2}{4t} \right) \right|.$$

For $z = x - w/N$, $x \in Q$ and $t \in I_0$, the arguments of Eg lie in a ball of radius $C_d(N^2 + |w|)$. The change of variables from (x, t) to those arguments is one-to-one, with absolute Jacobian comparable to λ^{d+2}. Since $\|g\|_p = \lambda^{-d/p}\|h\|_p$, (12.4) gives

$$\|S(t)h(x - w/N)\|_{L^p(Q\times I_0)} \lesssim \lambda^{d-(2d+2)/p}[\log(2 + N^2 + |w|)]^A\|h\|_p.$$

The power of λ is zero. Integrating against $|\check{\chi}(w)|\,\mathrm{d}w$ and using its rapid decay bounds the near contribution by $C[\log(2 + N)]^A\|f\|_{L^p(30Q)}$. Sum the pth powers over Q, using the bounded overlap of $30Q$, and add the far contribution. This proves (12.2) on I_0 with power A. On $[a/2, a]$ parabolic scaling gives an extra $a^{1/p}$ and replaces N by $\max(2, \sqrt{a}\,N)$. Sum the pth powers over dyadic intervals in a bounded time range and use conjugation for negative times.

For the mixed norms in (12.3), choose $p_c < a < p$ and take $b = a$ in (12.1). The diagonal case of [51, Corollary 1.2] takes such an $L^a \to L^a$ extension bound to the first inequality in (12.3) for every $p \le r \le \infty$. The second follows from spatial Littlewood–Paley theory and $\|\cdot\|_{\ell^p} \le \|\cdot\|_{\ell^2}$ for $p > 2$. □

In $d = 1$ this argument gives $(\log(2 + N))^{1/2}$ at $p_c = 4$. The sharp annular power on I_0 is $1/4$ [51, Theorem 1.4 and §4.1]. It is also the sharp power for a frequency ball: modulation moves $[-N, N]$ into $[N, 3N]$ and, at each time, only translates the output and multiplies it by a phase.

A The Dirichlet problem on a periodic fiber

The balance law in Section 4 uses a Dirichlet potential smooth up to a boundary that moves with the time variables. Section 5 bounds this potential by comparing determinants on the whole fiber and on small balls inside it. We prove the required existence, comparison, and smooth dependence here; the quantitative bounds in the wavelength h come from the comparison functions in Section 5. Alesker proved smooth Dirichlet solvability on quaternionic balls [3, Theorem 1.4]; Zhu proved it on bounded domains in quaternionic space in the presence of a smooth subsolution [81, Theorem 1.1]. Our fibers have periodic real directions and only three real coordinates per variable. Adjoining a fourth, periodic coordinate gives the local rotations needed to estimate mixed derivatives at the boundary.

Let $\Lambda \subset \mathbb{R}^m$ be a full lattice, $V \subset \mathbb{R}^{2m}$ a bounded domain with smooth boundary, and $\Omega = (\mathbb{R}^m/\Lambda) \times V$. The periodic variables occupy the real slots and the coordinates of V occupy the $\mathbf{i}, \mathbf{j}$ slots of the Hessian H from Lemma 3.1.

Theorem A.1 (Periodic Dirichlet problem)**.** *Let $m \ge 1$. Suppose $r \in C^\infty(\overline{V})$ is a defining function, negative in V, with $dr \ne 0$ on ∂V and $H(r) > 0$ on $\overline{\Omega}$. Let $B_0, g \in C^\infty(\overline{\Omega})$ be real and suppose*

$$H(B_0) \ge 0, \qquad g > \det H(B_0) \quad \text{on } \overline{\Omega}.$$

There is a unique smooth solution, up to the boundary, of

$$\det H(B_0 + u) = g, \qquad H(B_0 + u) > 0, \qquad u|_{\partial\Omega} = 0. \tag{A.1}$$

There are constants $b \ge b' > 0$ such that

$$br \le u \le b'r. \tag{A.2}$$

For smooth finite-dimensional families of such data and domains, these constants can be chosen locally uniformly. The solutions depend smoothly on the parameters up to the boundary, after any smooth local identification of the domains with a fixed domain.

Proof. Work on $\widetilde{\Omega} = \Omega \times (\mathbb{R}/\mathbb{Z})^m$, with the new coordinates in the $\mathbf{k}$ slots. Extend the data constantly in these coordinates. Until the last paragraph, G, H are the full quaternionic gradient and Hessian, which agree with the physical operators on the extended data. Write $\mathcal{H}(D^2 v) = H(v)$ and encode a real covector $\xi \in \mathbb{R}^{4m}$ as the quaternionic column $\widehat{\xi}$ in the slot basis. Lemma 3.1 gives $\mathcal{H}(\xi\xi^{\mathsf{t}}) = \widehat{\xi}\,\widehat{\xi}^*$ and $\operatorname{tr} H(v) = \Delta v$.

Comparison and continuity. Join the known solution $u = r$ to the given density by $g_t = (1-t)\det H(B_0 + r) + tg$, $0 \le t \le 1$. At a solution $\psi = B_0 + u$ put $Q = H(\psi)$ and $S = \operatorname{tr} Q^{-1}$. The derivative of the log determinant is

$$Lw = Q^{-1} : H(w), \qquad \sigma_L(\xi) = \widehat{\xi}^* Q^{-1} \widehat{\xi}.$$

Thus L is a uniformly elliptic real operator at each fixed smooth solution with $Q > 0$ on the closure. For two potentials ψ_1, ψ_2 with $Q_i = H(\psi_i) > 0$, the corresponding difference formula is

$$\log\det Q_1 - \log\det Q_2 = \left(\int_0^1 \left((1-\theta)Q_2 + \theta Q_1\right)^{-1} \mathrm{d}\theta \right) : H(\psi_1 - \psi_2).$$

When the Hessians are positive on the closure, the averaged operator is uniformly elliptic. If $\det Q_1 \ge \det Q_2$ on the domain and $\psi_1 \le \psi_2$ on its boundary, the maximum principle therefore gives $\psi_1 \le \psi_2$ throughout. This also gives uniqueness. The linear Dirichlet operator $L : C_0^{k+2,\alpha} \to C^{k,\alpha}$ is an isomorphism ($k \ge 0$, $0 < \alpha < 1$; the subscript denotes zero boundary trace). Indeed the local Schauder estimates and Dirichlet theory [31, Theorems 6.14 and 6.19] apply in a finite cover of the flat quotient. In the continuity method from Δ to L, r is a common strict barrier since $H(r) \ge cI$; the maximum principle therefore supplies the C^0 estimate. The starting Dirichlet Laplacian is invertible on the compact manifold with boundary. The implicit function theorem gives openness; the estimates below give closedness. Their constants may depend on the fiber and its data, but are uniform in t.

Both endpoints of the path have determinant strictly above $\det H(B_0)$ at every point. The gap is uniform by compactness. Hence we may choose $0 < b' < 1 < b$ so that

$$\det H(B_0 + b'r) < g_t < \det H(B_0 + br)$$

everywhere, using $H(r) \ge cI$ for the second inequality. Comparison gives (A.2). Since u vanishes on the boundary, $u = \alpha r$ in a collar with smooth quotient α at each smooth solution; division in the barriers gives $b' \le \alpha \le b$. In particular $\nabla u = \alpha \nabla r$ on the boundary. We also have $Lr \ge cS$ and $S \ge m g_t^{-1/m} \ge c_0 > 0$. For any constant real unit vector e, differentiation and concavity give

$$L\psi_e = (\log g_t)_e, \qquad L\psi_{ee} \ge (\log g_t)_{ee}. \tag{A.3}$$

The maximum principle applied to $\pm\psi_e + Cr$ bounds the full gradient. Applied to $\psi_{ee} + Cr$, it reduces an upper bound for the real Hessian to the boundary.

Second derivatives at the boundary. Center a flat chart at a boundary point, let ν be the outward unit normal, and identify vectors with their quaternionic columns in this chart. For real tangent vectors v, w, $D^2 u(v, w) = \alpha D^2 r(v, w)$ at the center. For the mixed derivatives we adapt the tangential differentiation argument of Caffarelli, Nirenberg and Spruck for the real Monge–Ampère equation [15, Section 2, (2.11)–(2.14)]. Fix a real tangent unit vector e. We construct a local quaternionic rigid motion $X = (e + Ax) \cdot \nabla$ for which Xr vanishes to second order along the boundary at the center. Let

$$v = \frac{P_{\tan} D^2 r\, e}{|\nabla r|}, \qquad A = v\nu^* - \nu v^* - \nu(\nu^* v)\nu^*,$$

where $P_{\tan}$ is the real orthogonal tangent projection. Since $\operatorname{Re}(\nu^* v) = 0$, we have $A^* = -A$ and $A\nu = v$. Now $Xr(0) = 0$ and for every tangent w,

$$\partial_w(Xr)(0) = D^2 r(w, e) - |\nabla r|\, w \cdot A\nu = 0.$$

Thus $|Xu| = |\alpha Xr| \le C|x|^2$ on the boundary near the center. The flow of X acts by quaternionic unitary rotations and translations. Differentiating the covariance of H under this flow gives $L(X\psi) = X \log g_t$: the commutator from conjugating Q has zero pairing with Q^{-1}. This identity uses only the flat chart. Subtracting the known XB_0 gives $|L(Xu)| \le CS$.

In a chart ball of fixed small radius, set $w = C_1|x|^2 - C_2 r$. First choose C_1 so that $w \ge |Xu|$ on the true boundary by the quadratic bound, and on the spherical boundary by the gradient bound. Then choose C_2 so that $L(w \pm Xu) \le 0$, using $H(|x|^2) = 8I$ and $Lr \ge cS$. The maximum principle gives $|Xu| \le w$ in the intersection. Both functions vanish at the center, so the inward derivative bounds $|\partial_\nu Xu|$. This is $|D^2 u(\nu, e)|$: the additional term $\nabla u \cdot A\nu$ vanishes since ∇u is normal. The constants are uniform over the compact boundary.

For the remaining derivative, the product rule at the boundary reads

$$H(u) = \alpha H(r) + G(\alpha)G(r)^* + G(r)G(\alpha)^*, \qquad G(r) = |\nabla r|\nu.$$

Compression to the *quaternionic* orthogonal complement of ν annihilates both cross terms. On that complement Q is therefore bounded below by $b' H(r)|_{\nu^\perp} \ge cI$. The rank-one rule shows that in a quaternionic orthonormal basis beginning with ν,

$$Q = \begin{pmatrix} \psi_{\nu\nu} + c_1 & z^* \\ z & D \end{pmatrix}.$$

Here c_1, z, D depend only on the real second derivatives already bounded. For $m > 1$ the Schur formula in Lemma 3.2 gives $\psi_{\nu\nu} = g_t / \det D + z^* D^{-1} z - c_1$, which is bounded since $D \ge cI$. For $m = 1$ the scalar identity $Q = g_t$ suffices. Equation (A.3) now gives $D^2\psi \le CI$ throughout the domain. Since $\Delta\psi = \operatorname{tr} Q > 0$, every real Hessian eigenvalue is at least $-(4m-1)C$. This is the full C^2 bound.

Real elliptic regularity. The C^2 bound and $\det Q = g_t \ge c_0 > 0$ put all eigenvalues of Q in a fixed interval $[c, C] \subset (0, \infty)$. Thus all real Hessians $D^2\psi$ lie in the compact convex set $\mathcal{K} = \{M : |M| \le C_1,\ cI \le \mathcal{H}(M) \le CI\}$. On $\mathcal{H}(M) > 0$ the smooth function $F(M) = \log\det \mathcal{H}(M)$ is concave. For $M \in \mathcal{K}$ and every real covector ξ,

$$DF(M)[\xi\xi^{\mathsf{t}}] = \widehat{\xi}^* \mathcal{H}(M)^{-1} \widehat{\xi} \in [C^{-1}|\xi|^2, c^{-1}|\xi|^2].$$

These are real uniform ellipticity and concavity on the range of the solution Hessians, the hypotheses (i)′, (ii)′ of the interior and boundary estimates [31, Theorems 17.14 and 17.26′]. To put F on the full real matrix space as in that theorem, multiply it by a smooth cutoff supported in $\{\mathcal{H}(M) > 0\}$ and equal to one on a neighborhood of $\mathcal{K}$. The equation $F(D^2\psi) - \log g_t = 0$ then has uniformly smooth coefficients and boundary data B_0. The cited local boundary estimates (see also Krylov [48]) and the interior Evans–Krylov estimate [22, 47] apply in a finite cover of the flat quotient, giving a uniform $C^{2,\alpha}$ bound for some $\alpha > 0$.

For a limit of solvable parameters, a subsequence converges in $C^{2,\beta}$, $0 < \beta < \alpha$. Positivity and the equation pass to the limit. Schauder bootstrapping gives smoothness up to the boundary: in flattened coordinates differentiate tangentially with the given Dirichlet data, and recover the pure normal derivative from the equation by uniform ellipticity. Thus the path is closed and reaches g.

Uniqueness on $\widetilde{\Omega}$ makes the solution invariant under translations in every added periodic coordinate. It descends to Ω; any physical solution lifts, so uniqueness descends as well. The

same lift gives comparison on subdomains, in particular on the balls in the three-slot fibers used in Section 5. For moving domains pull the lifted equation back to a fixed domain. Its coefficients and boundary data depend smoothly on the parameters, and its linearization is conjugate to the Dirichlet isomorphism above. The implicit function theorem in each $C_0^{k+2,\alpha}$ gives smooth dependence up to the boundary. The strict inequalities used to choose b, b' persist locally in the parameters. □

B The sharp logarithm with angular regularity

Write $p = p_c$, $d = n-1$, $\alpha = d/2$, $I = [1,2]$, and $L(v) = \log(2+v)$, so that $\alpha p = n$. Let $\Lambda_\omega = (1-\Delta_{S^{n-1}})^{1/2}$ act on the angular variable on each sphere about the origin. The joint weak spacetime norm is $\|u\|_{p,\infty} = \sup_{\lambda>0} \lambda|\{|u| > \lambda\}|^{1/p}$.

Theorem B.1 (The angular endpoint)**.** *Let $n \geq 2$, $p = p_c$, and $B > 1/2 + 1/p$. Then*

$$\|U(T)f\|_{L^p(\mathbb{R}^n \times I)} \leq C_{n,B}(\|L(J)^{1/p} f\|_p + \|L(\Lambda_\omega)^B f\|_p),$$
$$\|U(T)f\|_{L^{p,\infty}(\mathbb{R}^n \times I)} \leq C_{n,B}\|L(\Lambda_\omega)^B f\|_p.$$

In each inequality the angular norm on the right can be replaced by $\|\Lambda_\omega^\sigma f\|_p$ for any $\sigma > 0$, with a constant depending on σ.

For radial data the angular operator acts as a scalar, so Proposition 8.1 makes the frequency power $1/p$ in the strong estimate sharp. To prove the theorem we first separate spatial frequency and angular degree on a single annulus. Let $\Pi_N = \chi(|D|/N)$ with $\chi \in C_c^\infty((0,\infty))$. We use $A = 2/p$ for the power in Theorem 1.2; the argument applies with any positive exponent A for which that bound holds. An angular polynomial of degree at most M is a function whose restriction to almost every sphere about the origin is a sum of spherical harmonics of degrees at most M.

Proposition B.2 (Bounded angular degree on a frequency annulus)**.** *For $N \geq 2$ and an angular polynomial $f \in L^p(\mathbb{R}^n)$ of degree at most M,*

$$\|U(t)\Pi_N f\|_{L^p(I\times\mathbb{R}^n)} \lesssim [L(N)^{1/p} + L(M)^A]\|f\|_p, \tag{B.1}$$
$$\|U(t)\Pi_N f\|_{L^{p,\infty}(I\times\mathbb{R}^n)} \lesssim L(M)^A\|f\|_p.$$

The constants are independent of M and N.

The two terms in the strong estimate come from different output radii in physical space. When M is sufficiently small relative to N, we split the output at radius $(2+M)^K/N$ for a fixed K. The first lemma below shows that the logarithm on the inner ball counts its radius in wavelengths, here $(2+M)^K$, so the cost of observation is $O(L(M)^A)$. On its exterior the input from a small ball about the origin contributes a decaying error in the strong norm. For the remaining input, angular stationary phase at $x = r\theta$, $r \leq 4$, gives profiles $r^{-\alpha}G(t \pm r, \theta)$ with $\|G\|_{L^p(\,\mathrm{d}v\,\mathrm{d}\theta)} \lesssim \|f\|_p$, where $v = t \pm r$. Each dyadic interval of r contributes equally to the pth power of the radial factor, since $r^{d-\alpha p} = r^{-1}$. Its weak L^p norm, with the same polar measure $r^d\,\mathrm{d}r$, is bounded independently of the number of intervals.

For radial data, Müller and Seeger reduced the wave kernel to four one-dimensional kernels concentrated at $t \pm r \pm s = 0$, with s the input radius [58, Lemma 2.1]. Colzani, Cominardi and Stempak obtained the corresponding weak endpoint for the radial cosine propagator [18, Theorem 2.1]. We recover these four kernels outside the indicated ball with constants uniform in the angular degree.

B.1 Observation on a ball

Lemma B.3. *For $N \geq 2$, $0 < r \leq 1$ and every $x_0 \in \mathbb{R}^n$,*

$$\|U(t)\Pi_N f\|_{L^p(I\times B(x_0,r))} \lesssim \min\{1, (Nr)^{n/p}\}\, L(Nr)^A \|f\|_p.$$

Proof. Translate to $x_0 = 0$ and rescale to frequency one. With $T = N$, $R = Nr$, we must bound the norm on $[T, 2T] \times B(0, R)$ by $T^{1/p}\min(1, R^{n/p})L(R)^A\|f\|_p$. For the kernel of $U(t)\Pi_1$, write $v = |z|$ and $\langle v\rangle = (1 + v^2)^{1/2}$. Spherical stationary phase gives, for every D and $v \geq 1$,

$$K(t, z) = v^{-\alpha}k(t - v) + O_D\big(v^{-\alpha-1}\langle t - v\rangle^{-D} + v^{-\alpha}\langle t + v\rangle^{-D}\big). \tag{B.2}$$

Here k is the Fourier transform of a constant times $\rho^\alpha\chi(\rho)$ on the positive axis. After removal of the phases $e^{\pm iv\rho}$ the spherical amplitudes are symbols in $v\rho$; radial integration by parts gives the stated decay. For $v < 1$ the entire kernel is $O_D(\langle t\rangle^{-D})$. In particular, at any fixed x, Young's inequality in time on the input radii $T/2 \leq s = |y - x| \leq 3T$ gives

$$\|U(t)\Pi_1 f(x)\|_{L^p([T,2T])} \lesssim T^{1/p}\|f\|_p.$$

Indeed the convolution acts on $s^\alpha \int_{\mathbb{S}^d} |f(x + s\omega)|\,\mathrm{d}\omega$, whose pth power integrates to at most $CT\|f\|_p^p$ on these radii, since $\alpha p = d + 1$. At the remaining radii the kernel is $O_D((T + s)^{-D})$. Integration over x proves the assertion for $R \leq 2$.

Suppose $2 < R \leq T$ and put $S = 100R^{3+n/p}$. If $T \leq 10S$, the scaled global estimate of Theorem 1.2 suffices since $L(T) \lesssim L(R)$. For $T > 10S$, the input that reaches the observation ball at time Q occupies a shell of radius Q and thickness $O(R)$, up to rapidly decaying tails. We will compare its wave with one from a shell at radius S, where the global bound costs only $L(R)^A$. For this comparison take $Q \geq S$ and write the shell input as

$$f_Q(s\omega) = s^{-\alpha}h(s - Q, \omega), \qquad h(u, \omega) = 0 \quad (|u| > 5R).$$

Its contribution at $t = Q + b$, $|x|, |b| \leq R$, is approximated by

$$\mathcal{T}h(b, x) = \int k(b - u + x \cdot \omega)h(u, \omega)\,\mathrm{d}u\,\mathrm{d}\omega.$$

This expression is independent of Q. In fact $|s\omega - x| = s - x \cdot \omega + O(R^2/Q)$ and $(s/|s\omega - x|)^\alpha = 1 + O(R/Q)$. Inserting these into (B.2), with the polar Jacobian s^d, bounds the difference kernel against h by $C_D(R^2/Q)\langle b - u + x \cdot \omega\rangle^{-D}$. Young's inequality in b, followed by integration in ω and x, gives

$$\|U(Q + b)\Pi_1 f_Q - \mathcal{T}h\|_{L^p(\{|b|,|x|\leq R\})} \lesssim (R^{2+n/p}/Q)\|h\|_{L^p(\,\mathrm{d}u\,\mathrm{d}\omega)}.$$

We may therefore estimate the profile at the smaller radius $Q = S$. Since $d - \alpha p = -1$, this input has norm

$$\|f_S\|_p^p = \int \frac{|h(u, \omega)|^p}{S + u}\,\mathrm{d}u\,\mathrm{d}\omega \asymp S^{-1}\|h\|_p^p.$$

The factor $S^{1/p}$ in the scaled wave estimate cancels. Applying Theorem 1.2 on $[S/2, S]$ and $[S, 2S]$ and using $L(S) \lesssim L(R)$ now gives $\|\mathcal{T}h\|_{L^p(\{|b|,|x|\leq R\})} \lesssim L(R)^A\|h\|_p$.

Divide $[T, 2T]$ into intervals of length between $R/2$ and R, centered at Q_i. On interval i take $h_i(u, \omega) = (Q_i + u)^\alpha f((Q_i + u)\omega)\mathbf{1}_{|u|\leq 5R}$. The input shells have bounded overlap, so $\sum_i \|h_i\|_p^p \lesssim T\|f\|_p^p$. Applying the profile estimate and summing gives the desired bound for these shells. For $T/2 \leq s \leq 3T$, $|s - Q_i| > 5R$, the remaining kernel is bounded by $C_D s^{-\alpha}\mathbf{1}_{|t-s|>3R}\langle t - s\rangle^{-D}$: the output displacement is at most R. Young's inequality in t as above bounds this contribution by $C_D R^{n/p+1-D}T^{1/p}\|f\|_p$. The radii outside $[T/2, 3T]$ again give rapid decay in $T + s$. □

B.2 The polar kernel and the two endpoint norms

At an output point $r\theta$, angular stationary phase samples each input sphere at directions θ and $-\theta$. An angular average controls the remainders. To bound all three terms by a single radial profile, for g on $\mathbb{R}^n$ define the even function of $v \in \mathbb{R}$

$$h_g(v,\theta) = |v|^{d/p}\left(|g(|v|\theta)| + |g(-|v|\theta)| + \int_{\mathbb{S}^d} |g(|v|\omega)|\,\mathrm{d}\omega\right), \qquad G_g(v,\theta) = \mathcal{M}_{\mathbb{R}} h_g(v,\theta),$$

where $\mathcal{M}_{\mathbb{R}}$ is the Hardy–Littlewood maximal operator in v. The weight $|v|^{d/p}$ incorporates the polar Jacobian, so $\|G_g\|_{L^p(\,\mathrm{d}v\,\mathrm{d}\theta)} \lesssim \|g\|_p$.

Lemma B.4. *There is an integer K, independent of M, N, with the following property. Suppose g has angular degree at most M, is supported on $s = |y| \geq s_0$, and $N\min(r, s_0) \geq (2+M)^K$. Then for $t \in I$,*

$$|U(t)\Pi_N g(r\theta)| \lesssim \sum_{\pm} G_g(t \pm r, \theta)\begin{cases} r^{-\alpha}, & r \leq 4, \\ r^{-d/p}, & r > 4. \end{cases} \tag{B.3}$$

For arbitrary $g \in L^p(\mathbb{R}^n)$, the part supported in $|y| < N^{\eta-1}$, where $\eta = \alpha/(2n)$, satisfies

$$\|U(t)\Pi_N(\mathbf{1}_{|y|<N^{\eta-1}} g)\|_{L^p(I\times\mathbb{R}^n)} \lesssim N^{-\delta}\|g\|_p, \qquad \delta = (\alpha/2)(1-2/p) > 0. \tag{B.4}$$

Proof. Write $S_z F(\theta) = \int_{\mathbb{S}^d} e^{iz\theta\cdot\omega} F(\omega)\,\mathrm{d}\omega$. In polar coordinates the angular operator at radii r, s is

$$c_n s^d \int e^{-it\rho}\chi(\rho/N)\rho^d S_{r\rho} S_{-s\rho}\,\mathrm{d}\rho.$$

For each sign of z, use rotation-invariant cutoffs to separate the two stationary points. The expansion takes the form

$$S_z F = |z|^{-\alpha}\sum_{\sigma=\pm 1} e^{i\sigma|z|}(c_\sigma P_\sigma F + E_{\sigma,z}F),$$

where for either fixed sign of z the two P_σ are the identity and the antipodal reflection, in the corresponding order. Fix an integer $D > 2+\alpha$. Then $E_{\sigma,z}F$ and its first D logarithmic derivatives in $|z|$ are bounded pointwise by $C_D|z|^{-1}\|F\|_{C^{m_D}}$. To see this, use coordinates at the pole in which $1 \mp \theta\cdot\omega = |w|^2/2$. After removing the leading oscillation, the normalized integral has the form $I_\lambda(a) = \lambda^{d/2}\int e^{\pm i\lambda|w|^2/2} a(w)\,\mathrm{d}w$, where $\lambda = |z|$. Integration by parts in w gives $\lambda\partial_\lambda I_\lambda(a) = -\frac{1}{2} I_\lambda(w\cdot\nabla a)$. Each differentiated amplitude vanishes at the pole, so its leading stationary phase term is zero. The stationary phase remainder is $O(\lambda^{-1})$ for each of the finitely many amplitudes, with the stated derivative bound; off the pole one integrates by parts in angle [74, Chapter VIII, §2].

On degrees at most M, $\|F\|_2 \lesssim (2+M)^{d/2}\|F\|_1$ by the constant diagonal of the spherical projection kernel. Sobolev embedding and the spectrum of $-\Delta_{\mathbb{S}^d}$ thus bound $\|F\|_{C^{m_D}} \lesssim (2+M)^{c_D}\|F\|_1$. Choose $K > c_D$ for the fixed D above. For $|z| \gtrsim (2+M)^K$ each normalized remainder maps this band from L^1 to L^∞ with uniform norm. It preserves the band by rotation invariance. Consequently the composition of the two normalized factors, including the needed radial derivatives, is bounded at θ by $C(|F(\theta)| + |F(-\theta)| + \|F\|_1)$.

Now integrate by parts in ρ/N in each of the four phases. The scalar kernels against the last angular majorant are bounded by $C_D r^{-\alpha} s^\alpha N\langle N(t \pm r \pm s)\rangle^{-D}$. The input weight left after $s^{d/p}$ is $s^{1/p}$. For $r \leq 4$ it is bounded by $C\langle t \pm r \pm s\rangle^{1/p}$; for $r > 4$ insert the additional factor $r^{1/p}$ on the right. Absorb this polynomial into the decay. For each $v = t \pm r$, the integrals with phases $v - s$ and $v + s$ are bounded by one convolution on $\mathbb{R}$, since h_g is even in s. Its kernel is majorized by $C_D N\langle N\cdot\rangle^{-D+1/p}$, so the convolution is bounded by $CG_g(v,\theta)$. This proves (B.3).

Finally (B.2) and scaling give $\|U(t)\Pi_N\|_{1\to\infty} \lesssim N^{\alpha+1}$ for $t \in I$, while the L^2 norm is bounded. For $v = g\mathbf{1}_{|y|<b}$, interpolate $\|U(t)\Pi_N v\|_2$ and $\|U(t)\Pi_N v\|_\infty$, then use Hölder on the input ball. This gives $\|U(t)\Pi_N v\|_p \lesssim (N^{\alpha+1}b^n)^{1-2/p}\|g\|_p$. Taking $b = N^{\eta-1}$ gives (B.4). □

Proof of Proposition B.2. Put $H = (2+M)^K$. If $H > N^\eta$, then $L(N) \lesssim L(M)$ and the unrestricted annular estimate gives both conclusions. Suppose $H \le N^\eta$. On the output ball $r \le H/N$, Lemma B.3 gives the strong bound $CL(H)^A\|f\|_p \lesssim L(M)^A\|f\|_p$. On its complement we may discard the input from $s < N^{\eta-1}$ with the strong error (B.4). For the remaining input g, both Nr and Ns are at least H, as required in Lemma B.4.

Consider one profile in (B.3). Changing variables from (t, r) to $(v, r) = (t \pm r, r)$ preserves measure. At fixed v, θ, write $G = G_g(v, \theta)$. For $N^{-1} \le r \le 4$ the strong norm and the distribution function are controlled by the two radial integrals

$$\int_{N^{-1}}^4 (r^{-\alpha}G)^p r^d \,\mathrm{d}r = G^p \log(4N), \qquad \int_{\{r>0:\ r^{-\alpha}G>\lambda\}} r^d \,\mathrm{d}r = \frac{G^p}{n\lambda^p} \quad (\lambda > 0). \tag{B.5}$$

Integrating in v, θ gives the factor $O(L(N)^{1/p})$ for the strong norm and a constant for the joint weak norm. For $r > 4$ the pth power of $r^{-d/p}$ cancels the polar Jacobian. At each t the remaining integral of $|G_g(t \pm r, \theta)|^p$ in r, θ is at most $\|G_g\|_p^p$; this part has a strong bound uniform in N. Combining the two regions proves the proposition. □

For $A = 2/p$ and $M \le \exp(\sqrt{L(N)})$, the strong estimate therefore costs only $L(N)^{1/p}$. For cutoffs equal to one on $[5/4, 7/4]$, the radial data of Proposition 8.1 force this power already at angular degree zero.

B.3 Summation over the spatial and angular frequencies

Choose a smooth radial Littlewood–Paley decomposition f_j at frequencies $N_j = 2^j$, and let $u_j = U(t)f_j$. Insert a fixed enlarged annular multiplier Π_{N_j} equal to one on the support of $\widehat{f_j}$. The bounded frequency part satisfies a uniform fixed-time L^p estimate by the Mikhlin theorem. On the other bands, (B.4) allows us to restrict the input to $s \ge N_j^{\eta-1}$, with a total strong error $O(\|f\|_p)$ even when summed in j.

We group angular degrees according to the size of their logarithm. Use smooth angular low pass operators S_m at degree $M_m = 2^{2^m}$, $m \ge 0$, and their differences $Q_m = S_m - S_{m-1}$ with $S_{-1} = 0$. Take a fixed smooth cutoff on degrees, equal to one up to M_m and zero beyond $2M_m$. These operators commute with all radial Fourier multipliers by rotation invariance. Their kernels satisfy $|S_m(\theta, \omega)| \lesssim_D M_m^d(1 + M_m \operatorname{dist}(\theta, \omega))^{-D}$ by smooth spherical localization [60, Theorem 3.5 and Corollaries 3.6, 5.3]; on the circle this is ordinary smooth Fourier summation. In particular $|S_m g| \lesssim \mathcal{M}_{\mathbb{S}^d} g$, uniformly in m. For every $B > 0$,

$$\|Q_m f\|_p \lesssim_B 2^{-mB}\|L(\Lambda_\omega)^B f\|_p. \tag{B.6}$$

Indeed, as a function of the harmonic degree ν on the support of Q_m, $L((1 + \nu(\nu + d - 1))^{1/2}) \asymp 2^m$. The normalized symbol $2^{mB}Q_m(\nu)L((1 + \nu(\nu + d - 1))^{1/2})^{-B}$ and all its logarithmic derivatives in ν are uniformly bounded. The smooth multiplier theorem on the sphere [11, Theorem 4.9(1)] gives (B.6) on each angular slice.

To pay the spatial logarithm only once, at fixed r and N_j we group all angular bands on which stationary phase is uniform into one low pass. Put $H_m = (2 + 2M_m)^K$, and select the largest m with

$$H_m \le \min(N_j^\eta, N_j r). \tag{B.7}$$

Call this index $m(j,r)$, with value -1 when there is no such m. For $a_j = \mathbf{1}_{s\ge N_j^{\eta-1}} f_j$, the exact decomposition of the retained wave at $(t, r\theta)$ is

$$U(t)\Pi_{N_j} a_j = U(t)\Pi_{N_j} S_{m(j,r)} a_j + \sum_{m>m(j,r)} U(t)\Pi_{N_j} Q_m a_j.$$

For a given output r, the index $m(j,r)$ is fixed on every input sphere. If it is nonnegative, Lemma B.4 applies to the first term. Its positive majorant may use $\mathcal{M}_{\mathbb{S}^d} f_j$ in place of $S_{m(j,r)} f_j$, eliminating the dependence of the profile on the chosen cutoff. The vector-valued maximal inequality, on the sphere and on the line [33, Theorem 1.2], gives for arbitrary nonnegative coefficients c_j

$$\|(c_j G_{\mathcal{M}_{\mathbb{S}^d} f_j})_j\|_{L^p(\,\mathrm{d}v\,\mathrm{d}\theta;\ell^2)} \lesssim \|(c_j f_j)_j\|_{L^p(\mathbb{R}^n;\ell^2)}.$$

The reflection and angular integral in h_g are bounded on $L^p(\mathbb{S}^d;\ell^2)$ by Minkowski's inequality.

It remains to integrate these profiles along $v = t \pm r$, now taking the square function in j. Write $G_j = G_{\mathcal{M}_{\mathbb{S}^d} f_j}(v,\theta)$. On the selected region $r \ge N_j^{-1}$, so for $r \le 4$ we may bound $r^{-\alpha}$ by $C(r + N_j^{-1})^{-\alpha}$. At fixed v, θ, Minkowski's inequality in $L^{p/2}(r^d\,\mathrm{d}r)$ gives

$$\left\| \left(\sum_j (r + N_j^{-1})^{-2\alpha} |G_j|^2 \right)^{1/2} \right\|_{L^p((0,4),r^d\,\mathrm{d}r)} \lesssim \left(\sum_j (1+j)^{2/p} |G_j|^2 \right)^{1/2}. \tag{B.8}$$

Indeed the pth power of the jth radial factor integrates to $O(1+j)$ by $\alpha p = n$. Integrate in v, θ and apply the maximal inequality above with $c_j = (1+j)^{1/p}$. The smooth spatial Littlewood–Paley theorem then bounds the strong square function by $C\|L(J)^{1/p} f\|_p$. For the weak norm keep the common factor $r^{-\alpha}$ and use the distribution formula in (B.5) with $G = \|(G_j)\|_{\ell^2}$. The maximal inequality with $c_j = 1$ gives $C\|f\|_p$. For $r > 4$ the common factor is $r^{-d/p}$; the same integration at fixed t as in the single annulus gives a strong bound of $C\|f\|_p$ there.

It remains to estimate the unselected angular frequencies in strong L^p. Fix m and denote by v_j the wave $U(t)\Pi_{N_j}$ applied to $Q_m f_j \mathbf{1}_{s\ge N_j^{\eta-1}}$; write $b_j = \|Q_m f_j\|_p$. When $H_m > N_j^\eta$, the entire v_j is unselected. There are $O(1 + \log H_m)$ such j and Theorem 1.2 bounds each by $CL(H_m)^A b_j$. The pointwise inequality $\|w\|_{\ell^2} \le (\#j)^{1/2-1/p}\|w\|_{\ell^p}$ shows that their strong square function costs $CL(H_m)^{A+1/2-1/p}\|(b_j)\|_{\ell^p}$.

For the remaining j, the unselected output lies in $r < R_j = H_m/N_j$. Let $D = \lceil \log_2(2H_m) \rceil$ and consider the disjoint shells $R_{k+1} < r \le R_k$, with $R_k = H_m 2^{-k}$. On shell k only $j \le k$ contribute. For $k - D \le j \le k$ the same finite-sequence inequality costs $(1+D)^{1/2-1/p}$. After taking pth powers and summing over k, each v_j is integrated only on its ball $r \le R_j$, where Lemma B.3 gives $CL(H_m)^A b_j$. For $j < k - D$ the shell lies within one wavelength: $N_j R_k = H_m 2^{j-k} \le 2^{-(k-j-D)}$. The volume factor in the same lemma on the smaller ball $r \le R_k$ gives $C2^{-(k-j-D)n/p} b_j$. Sum these latter bounds first in j by the triangle inequality and then in $\ell^p(k)$ by discrete Young's inequality. Thus the unselected mth angular band has total strong square-function norm at most

$$CL(H_m)^{A+1/2-1/p}\|(b_j)\|_{\ell^p} \lesssim 2^{m(A+1/2-1/p)}\|Q_m f\|_p.$$

In the last step we used $p > 2$ and the spatial square-function bound for $Q_m f$. By (B.6), summing in m costs $C_B\|L(\Lambda_\omega)^B f\|_p$ whenever $B > A + 1/2 - 1/p$.

These estimates control the square function of the original u_j: the unselected pieces have a strong bound, and the selected pieces have both bounds. The splitting depends on r, so the pieces need not be annular separately. We recover the sum only after reassembling u_j: the spatial synthesis

operator $(w_j) \mapsto \sum_j \Pi_{N_j} w_j$ is bounded from $L^q(I \times \mathbb{R}^n; \ell^2)$ to $L^q(I \times \mathbb{R}^n)$ for $1 < q < \infty$ by spatial Littlewood–Paley duality and Fubini [32, Theorem 6.1.2 and Remark 6.1.3]. Real interpolation gives the same assertion for joint weak L^p. The argument first gives uniform bounds for finite frequency sums and then the stated bounds by approximation. The same angular multiplier theorem applies to $L(\Lambda_\omega)^{-B}$ and $L(\Lambda_\omega)^B \Lambda_\omega^{-\varepsilon}$ for every $\varepsilon > 0$: the first absorbs $\|f\|_p$ into the stated weak norm, and the second gives the assertion with a positive angular derivative. This proves Theorem B.1.

C Disjoint packets and the weak spacetime norm

We prove Proposition 8.2. The geometry is the one used in Fefferman's ball multiplier example [23]; Wolff explains its relation to local smoothing in the discussion of (39) in [79]. For the quantitative compression factor $(\log(1/\delta))^{-(n-1)}$ with disjoint translates of unit tubes, see also Fernández-Delgado and de la Salle [24, Theorem 1.4]. We give the short construction needed here. The tubes are disjoint on one longitudinal interval. On a second interval, their union is small and the sum of their individual volumes stays bounded below. A modulated packet travels from the first part of its tube to the second.

An arrangement of tubes. Write $x = (X, Y) \in \mathbb{R} \times \mathbb{R}^d$, $d = n - 1$. Let $k = \lfloor \frac{1}{4} \log_2 N \rfloor$, $\delta = 2^{-k}$, and put $\alpha = 1/(4\sqrt{d})$, $w = \alpha\delta/16$. For $0 \le j < 2^k$, write $j\delta = \sum_{\ell=1}^{k} \varepsilon_\ell(j) 2^{-\ell}$ in binary. Use the digits to choose the intercept of a line of slope $s_j = \alpha j \delta$:

$$a_\ell = 1 - \ell/k, \qquad L_j(X) = \alpha \sum_{\ell=1}^{k} \varepsilon_\ell(j) 2^{-\ell} (X - a_\ell).$$

The ℓth binary increment changes the slope without moving the line at $X = a_\ell$. Near this point its displacement is small. At $X \le -1/2$ these lines are separated. Indeed, when one passes from j to $j + 1$, one digit, say m, changes from zero to one, and the later digits change from one to zero. Hence

$$L_{j+1}(X) - L_j(X) = \alpha\delta(X - a_m) + \alpha \sum_{\ell > m} 2^{-\ell}(a_\ell - a_m) \le -\alpha\delta/2.$$

The intervals of radius w around the $L_j(X)$ are therefore disjoint.

For X in $J = [1/4, 1/2]$, let B_X be the union of intervals of radius $w/2$ around the same $L_j(X)$. To see why this union is small, freeze the first $m = \lfloor k(1 - X) \rfloor$ digits. For $\ell > m$ we have $0 \le X - a_\ell \le (\ell - m)/k$. The tail of each line therefore ranges over an interval of length at most $\alpha \sum_{\ell > m} 2^{-\ell}(\ell - m)/k \le 2\alpha 2^{-m}/k$. There are 2^m choices for the frozen digits, and $m \le 3k/4$, so

$$|B_X| \le 2\alpha/k + 2^m w \lesssim \alpha/k.$$

Taking an independent choice in each transverse coordinate produces δ^{-d} lines, indexed by $\nu = (j_1, \dots, j_d)$. Write $L_\nu = (L_{j_1}, \dots, L_{j_d})$ and $s_\nu = (s_{j_1}, \dots, s_{j_d})$. The output cores

$$S_\nu = \{(X, Y) : X \in J,\ |Y_i - L_{j_i}(X)| \le w/2,\ 1 \le i \le d\}$$

all lie in $E = \{(X, Y) : X \in J,\ Y \in (B_X)^d\}$. We have proved

$$|E| \lesssim_n k^{-d}, \qquad \sum_\nu |S_\nu| = \delta^{-d}|J| w^d \asymp_n 1. \tag{C.1}$$

Transport at frequency N. Choose $0 \le \chi, \theta \le 1$ smooth, with χ supported in $(-2, -1/2)$ and equal to one on $[-3/2, -3/4]$, and θ supported in $(-1, 1)$ and equal to one on $[-1/2, 1/2]$. Define

$$b_\nu(X, Y) = \chi(X) \prod_{i=1}^{d} \theta((Y_i - L_{j_i}(X))/w), \qquad v_\nu = (1, s_\nu)(1 + |s_\nu|^2)^{-1/2}.$$

The b_ν have disjoint supports and their total support volume is $O_n(\delta^{-d} w^d) = O_n(1)$. Set $\lambda = 3N/2$, $f_\nu(x) = e^{i\lambda v_\nu \cdot x} b_\nu(x)$, and let $Q_N = \beta(|D|/N)$, where $\beta \in C_c^\infty((5/4, 7/4))$ is one on a neighborhood of $3/2$. We claim that, uniformly in ν, x and $1 \le t \le 2$,

$$U(t) Q_N f_\nu(x) = e^{i\lambda(v_\nu \cdot x - t)}\, b_\nu(x - t v_\nu) + O_n(N^{-1} w^{-2}). \tag{C.2}$$

To verify this, write $\xi = \lambda v_\nu + \eta$. On $|\eta| \le cN$ the cutoff is one and $|\xi| = \lambda + v_\nu \cdot \eta + O(|\eta|^2/N)$. The absolute Fourier transform of the bump is

$$w^d |\widehat{\chi}(\eta_1 + s_\nu \cdot \eta')| \prod_{i=1}^{d} |\widehat{\theta}(w \eta_i')|, \qquad \eta = (\eta_1, \eta').$$

The change of variables in these arguments and $|s_\nu| \le 1/4$ give $\int |\eta|^2 |\widehat{b}_\nu(\eta)|\, \mathrm{d}\eta \lesssim_n w^{-2}$. Thus the integrated phase error is $O_n(N^{-1} w^{-2})$; the absolute integral over $|\eta| > cN$ is $O_n(N^{-2} w^{-2})$. This proves (C.2). Its error tends to zero since $w \asymp_n \delta$ and $N \ge \delta^{-4}$.

Take $I_0 = [3/2, 7/4]$. Translation by $t v_\nu$ preserves each transverse difference $Y_i - L_{j_i}(X)$. Moreover, for $X \in J$ and $t \in I_0$, we have $-3/2 \le X - t(v_\nu)_1 \le -3/4$, since $4/\sqrt{17} \le (v_\nu)_1 \le 1$. Thus $b_\nu(x - t v_\nu) = 1$ on $S_\nu \times I_0$ and $|U(t) Q_N f_\nu| \ge 1/2$ there for large N.

Choosing the signs. For independent random signs ϵ_ν put $f = \sum_\nu \epsilon_\nu f_\nu$. The disjoint initial supports give $\|f\|_p \lesssim_{n,p} 1$ for every choice of signs. Independence and (C.1) give

$$\mathbb{E} \int_{E \times I_0} |U(t) Q_N f(x)|^2 \, \mathrm{d}x \, \mathrm{d}t = \sum_\nu \int_{E \times I_0} |U(t) Q_N f_\nu(x)|^2 \, \mathrm{d}x \, \mathrm{d}t \ge \tfrac{1}{4} |I_0| \sum_\nu |S_\nu| \gtrsim_n 1.$$

Fix signs attaining this lower bound and set $g_N = Q_N f$. Its Fourier support is as stated, and $\|g_N\|_p \lesssim_{n,p} 1$ by the uniform L^1 kernel bound for Q_N. For $p > 2$ and a measurable set H of finite measure, integration of the decreasing rearrangement gives

$$\int_H |u|^2 \le \frac{p}{p-2} |H|^{1-2/p} \|u\|_{L^{p,\infty}}^2.$$

Taking $H = E \times I_0$ and using $|E| \lesssim_n k^{-d}$ now gives $\|U(t) g_N\|_{L^{p,\infty}} \gtrsim_{n,p} k^{d(1/2 - 1/p)}$. As $k \asymp \log N$, this proves Proposition 8.2. In fact the inputs have bounded L^2 and L^∞ norms separately, so interpolation also gives a bounded Lorentz $L^{p,1}$ input norm.

These packets also test the angular weight in Theorem B.1. At $p = p_c$ a weak bound with only $[\log(2 + \Lambda_\omega)]^B$ on the input requires $B \ge 1/p$. Indeed, before the radial projection the input is supported in a fixed ball and is a sum of disjoint packets. Its first and second spatial derivatives are bounded by CN and CN^2, respectively, since $w^{-1} \lesssim_n N^{1/4}$. The spherical Laplacian is a sum of squares of infinitesimal rotations $x_i \partial_j - x_j \partial_i$. It follows that $\|(1 + \Lambda_\omega^2/N^2) g_N\|_p \lesssim 1$, since the radial projection commutes with rotations. For $B \ge 0$ the spherical multiplier $[\log(2 + \Lambda_\omega)]^B / (1 + \Lambda_\omega^2/N^2)$ has norm $O_B((\log(2 + N))^B)$ by the multiplier bound used in (B.6); for $B < 0$ the logarithmic weight itself is bounded. The preceding lower bound forces $B \ge 1/p$. Theorem B.1 gives sufficiency for $B > 1/2 + 1/p$.

D A linear Dirichlet problem for the planar wave

When $n = 2$, the coefficient of a rank-one background in the spatial determinant is linear in the unknown Hessian. We can therefore prove the $p_c = 4$ case of Theorem 1.2 with a linear auxiliary Dirichlet problem on a disk. We give the reduction, including the boundary estimate needed to apply Lemma 6.1.

The disk equation. Fix nonzero periodic data in a single cap as in Proposition 5.1. Retain $z = (T, \tau)$, $s = |\tau|$, $a = Kh$, and $Q = \Phi/a$. For $a < s < 4$ replace the ellipsoid by the product

$$\Omega_z^{\mathrm{pl}} : \quad r_W = |W|^2/w_0^2 - 1 < 0, \qquad |Q| < 1/10.$$

The real variable X still runs over the fixed torus. At fixed τ, (4.2) gives a holomorphic chart from this product to the imaginary spatial variables (P, Z). Use the wave part $A = A_{\mathrm{w}} = g(Q)(J_h *_X A^0)$ of (4.5), with $g = 1$ on $|Q| \le 1/100$ and $g = 0$ on $|Q| \ge 1/20$, and put $\mu = A_{TT}$. We need only the nonnegativity of this density; no regularizing cofactor is needed.

Put $B = H(|\Phi|^2)$, let B_{fib} be its spatial block, and write $L_B u = C(B_{\mathrm{fib}}) : H_{\mathrm{fib}}(u)$. With the row $c = (\partial_P \Phi, \partial_Z \Phi)$ and the column $e = (c_2, -c_1)^{\mathsf{t}}$, we have $B_{\mathrm{fib}} = 4c^* c$ and $C(B_{\mathrm{fib}}) = 4ee^*$. Define $\mathcal{J} = \det_{\mathbb{C}} \partial(W, \Phi)/\partial(P, Z) = (1 + W^2/2)/(\sqrt{2}Y)$. Then

$$L_B u = 4 \sum_{i,j=1}^{2} \mathrm{Re}(e_i \bar{e}_j) \partial_{X_i X_j} u + 4|\mathcal{J}|^2 \Delta_W u, \qquad |\mathcal{J}| \asymp s^{-1}. \tag{D.1}$$

Indeed, the mixed derivatives between X and (P, Z) in the Hessian take values in $\mathbb{C}_{\mathbf{k}}\mathbf{i}$, orthogonal to the entries of ee^* in $\mathbb{C}_{\mathbf{k}}$. The remaining imaginary block is the complex Hessian of Lemma 3.1; under the holomorphic chart it pairs with the vector $(\partial W\, e, ce) = (\mathcal{J}, 0)$. The Jacobian bound follows from (4.2).

Solve $L_B u = \mu$ with $u = 0$ on $|W| = w_0$. At each fixed (z, Φ) and real spatial Fourier frequency k this means solving

$$4|\mathcal{J}|^2 \Delta_W \widehat{u} - 4|k \cdot e|^2 \widehat{u} = \widehat{\mu} \quad (|W| < w_0), \qquad \widehat{u}|_{|W| = w_0} = 0.$$

Here Δ_W is the ordinary planar Laplacian. After division by $4|\mathcal{J}|^2$ the operator is the Dirichlet Laplacian minus a smooth nonnegative potential. It is invertible also at $k = 0$. The Dirichlet isomorphism on Hölder spaces and the implicit function theorem give smooth dependence on the parameters. There are only finitely many frequencies in μ. Solving before multiplication by $g(Q)$ shows that $u = g(Q)\widetilde{u}$ with $\widetilde{u}$ smooth and real. Thus u vanishes near the Q wall, including the corners. Also $u \le 0$: for every $\epsilon > 0$, $L_B(u + \epsilon r_W) = \mu + 16\epsilon|\mathcal{J}|^2/w_0^2 > 0$, ruling out a positive interior maximum in (X, W). Now let $\epsilon \downarrow 0$.

The energy. For the full three-by-three Hessian $H = H(u)$, let $\mathcal{D}_B(H) = B : C(H)$, the coefficient of λ in $\det(\lambda B + H)$. It is quadratic in H. Its derivative M in H is a mixed cofactor, hence divergence free by Lemma 3.3. Homogeneity gives $M : H = 2\mathcal{D}_B(H)$; differentiation in the TT entry gives $M_{TT} = C(B_{\mathrm{fib}}) : H_{\mathrm{fib}}(u) = \mu$. On the W wall, the full $H(r_W)$ has image the quaternionic line spanned by its normal $\nu = G(r_W)/|\nabla r_W|$. Since $u = 0$ there, writing $u = (u/r_W) r_W$ and restricting the product rule to $\nu^{\perp}$ gives $H|_{\nu^{\perp}} = 0$. The coefficient of λ in $\det((\lambda B + H)|_{\nu^{\perp}})$ is therefore zero, so $\nu^* M \nu = 0$ by Lemma 3.2. Now set

$$X_0(z) = \int_{\Omega_z^{\mathrm{pl}}} (-u)\mu \, \mathrm{d}\zeta, \qquad \mathcal{F}_{\mathrm{pl}}(z) = \tfrac{1}{2} X_0(z), \qquad \mathrm{d}\zeta = \mathrm{d}X \, \mathrm{d}P \, \mathrm{d}Z.$$

Green's formula (4.13) for M gives $\Delta_z X_0 = -2\int \mathcal{D}_B(H)\,\mathrm{d}\zeta$; the Q wall contributes nothing because u vanishes in a neighborhood of it.

To apply the same formula to A, we claim that $A : H \geq \mathcal{D}_B(H)$. The current is supported on the quaternionic plane $E = \ker B$ by Proposition 2.2. Put $\beta = \operatorname{tr} B$, $\beta_f = \operatorname{tr} B_{\mathrm{fib}}$. In E take the orthonormal basis consisting of the normalized projection of the time vector e_T and the spatial vector $(0, e/|e|)$. In this basis write $\alpha = A|_E$, $b = H|_E$, indexing rows from zero. Before normalization the projection of e_T has squared length β_f/β. Hence $\alpha_{00}\beta_f/\beta = A_{TT} = \mu = \beta_f b_{11}$ by the disk equation. The rank-one determinant identity gives $\mathcal{D}_B(H) = \beta \det b$. Substituting $\alpha_{00} = \beta b_{11}$ and completing the square gives

$$A : H - \mathcal{D}_B(H) = \beta\,|b_{01} + \alpha_{01}/\beta|^2 + \det(\alpha)/\beta \geq 0.$$

Green's formula for A now bounds $\Delta_z X_0$ by $\frac{1}{2}\Delta_z X_0$ plus the outward wave flux. Consequently

$$\mathcal{F}_{\mathrm{pl}} \geq 0, \qquad \Delta_z \mathcal{F}_{\mathrm{pl}} \leq \int_{|Q|<1/10} \delta(r_W)(u/r_W)G(r_W)^* A G(r_W)\,\mathrm{d}\zeta. \tag{D.2}$$

Comparison on the product fiber. Write $\sigma = s/h$, $g_T = e^{-iT|D|}f$, and $P_* = J_{Ds} * |g_T|^2$ for D sufficiently large. The multiplier proof of (5.5) applies throughout the product: $y \asymp s$ there and $|W| = w_0$ on the W wall. It gives $\|A\| \lesssim s^{-2}\sigma^m P_*$, with the additional factor $e^{-c_0\sigma}$ on that wall. By (D.1) and (5.4),

$$L_B(r_W P_*) = 16|\mathcal{J}|^2 P_*/w_0^2 + r_W L_B P_* \geq (c - CD^{-2})s^{-2}P_*.$$

The two variables separate, so no mixed gradient term occurs. For fixed large C_1, m_1 this gives $L_B(C_1\sigma^{m_1} r_W P_*) > \mu$. The comparison function and u both vanish on the W wall. Comparison on each fixed Q slice yields $0 \leq u/r_W \leq C_1\sigma^{m_1}P_*$ up to that wall. The original volume is $\mathrm{d}P\,\mathrm{d}Z = a^2|\mathcal{J}|^{-2}\,\mathrm{d}W\,\mathrm{d}Q$, while $\int \delta(r_W)\,\mathrm{d}W = \pi w_0^2$ and the full $|G(r_W)| \lesssim s^{-1}$. Integrating the energy and (D.2), using $\int P_*^2\,\mathrm{d}X \leq \|g_T\|_4^4$, gives

$$\mathcal{F}_{\mathrm{pl}} \lesssim a^2\sigma^m \|g_T\|_4^4, \qquad \Delta_z \mathcal{F}_{\mathrm{pl}} \lesssim (a/s)^2\sigma^m e^{-c_0\sigma}\|g_T\|_4^4.$$

For the lower bound use the centers $|W| \leq \gamma$, $|Q| \leq 1/400$ of Section 5. The inverse multiplier in the proof of (5.11) bounds the wave density itself below by m_X there, on the entire physical spatial ball of radius h about each center; take $d = 1$ in that formula. For the distance l to its center, $H_{\mathrm{fib}}(l^2) = 6I$, so $L_B(l^2) = 24|e|^2 \leq C$. For $m_X > 0$, subtract $\epsilon m_X(l^2 - h^2)$ from u, with ϵ small. The difference has strictly positive L_B on the whole ball and is nonpositive on its boundary, so it cannot have a positive interior maximum. This gives $(-u)\mu \geq \epsilon h^2 m_X^2$ at the center, also trivially for $m_X = 0$. Its integral gives

$$\mathcal{F}_{\mathrm{pl}} \gtrsim_\gamma a^2\sigma^{-m}e^{-C\gamma^2\sigma}\|g_T\|_4^4,$$

since the imaginary center region has volume comparable to $a^2s^2\gamma^2$ and $h^2s^2(hs)^{-2} = 1$. Taking γ small and then K large gives all three inequalities of Proposition 5.1 for $\mathcal{F}_{\mathrm{pl}}$. Lemma 6.1 and Section 7 now give the planar $O((\log(2/h))^{1/2})$ estimate.

E Curved characteristic sheets of symmetric systems

In the proof of Theorem 1.2, the wave symbol supplies both a polarization for each plane wave and a Gaussian centered at that polarization. A curved sheet of the largest eigenvalue of a symmetric matrix pencil supplies the same two objects. We construct them on one angular patch and check the two places where the phase enters the fiber estimate: Gaussian decay on the wall and recovery of the wave near the center.

Theorem E.1. *Let $C_1, \dots, C_n$ be real symmetric $n \times n$ matrices, $n \geq 2$, and put $C(\xi) = \sum_{j=1}^{n} \xi_j C_j$ and $\phi(\xi) = \lambda_{\max}(C(\xi))$. Suppose that on an open cone $\Gamma \subset \mathbb{R}^n \setminus \{0\}$ this eigenvalue is simple and $D^2\phi(\xi)$ is positive definite on $\xi^{\perp}$. For every $b \in C_c^{\infty}(\Gamma)$, $0 < h \leq 1$ and Schwartz function f on $\mathbb{R}^n$,*

$$\|e^{-\mathrm{i}T\phi(D)} b(hD) f\|_{L^{p_c}(\mathbb{R}^n \times [1,2])} \leq C_b [\log(2/h)]^{2/p_c} \|f\|_{L^{p_c}(\mathbb{R}^n)}, \qquad p_c = \frac{2n}{n-1}. \tag{E.1}$$

The constant may depend on the pencil. The same assertion holds for a simple smallest eigenvalue with negative definite transverse Hessian.

Example E.2 (A cubic characteristic sheet)**.** In three spatial dimensions take the pencil

$$C(x, y, z) = \begin{pmatrix} z & x & y \\ x & -z & 0 \\ y & 0 & -2z \end{pmatrix}.$$

At $(0, 0, 1)$ the top eigenvalue is 1, and its transverse Hessian is $\mathrm{diag}(1, 2/3)$ by (E.2) below. Thus (E.1) gives an L^3 bound on a neighborhood of this ray. The sheet is not a piece of a quadric. Indeed its characteristic polynomial is

$$(t - z)(t + z)(t + 2z) - x^2(t + 2z) - y^2(t + z).$$

If this homogeneous cubic factored over $\mathbb{C}$, it would have a linear divisor. Since it is monic in t, this entails a root $t = \alpha x + \beta y + \delta z$. At $z = 0$ the polynomial is $t(t^2 - x^2 - y^2)$. Since $x^2 + y^2$ is not the square of a linear form, $\alpha = \beta = 0$; the coefficients of $x^2 z$ and $y^2 z$ would then require $\delta = -2$ and $\delta = -1$, respectively. Irreducibility rules out even a local quadratic equation for the sheet: the resultant with that equation in t would vanish on an open set in (x, y, z), and hence identically.

E.1 Eigenvectors and the spatial fiber

Fix a unit direction $\omega \in \Gamma$. Take the unit top eigenvector of $C(\omega)$ as the first state coordinate e_0. The other state coordinates form its orthogonal complement of dimension $d = n - 1$. The isolated eigenprojection is smooth near ω by its resolvent formula [43, Chapter II]. Applying it to e_0 and normalizing the first component to one gives a real eigenvector $E(\xi) = (1, v(\xi))^{\mathsf{t}}$, homogeneous of degree zero. Put $S(\xi) = \phi(\xi) I - C(\xi)$ and denote its lower $d \times d$ block by $S_{\perp}(\xi)$. On a sufficiently small conic neighborhood,

$$S(\xi) E(\xi) = 0, \qquad c|\xi| I \leq S_{\perp}(\xi) \leq C|\xi| I.$$

Here and below constants are uniform on each chosen compact angular chart.

Write Lw for the orthogonal projection of $C(w)e_0$ on $e_0^{\perp}$. Differentiating the unit eigenvector equation along $\omega + \sigma w$ shows that the derivative of the eigenvector at $\sigma = 0$ is $S_{\perp}(\omega)^{-1} Lw$ in $e_0^{\perp}$; differentiating the eigenvalue derivative $\phi'_w = e^{\mathsf{t}} C(w) e$ therefore gives

$$D^2\phi(\omega)[w, w] = 2(Lw)^{\mathsf{t}} S_{\perp}(\omega)^{-1} Lw. \tag{E.2}$$

Consequently $L : \omega^{\perp} \to e_0^{\perp}$ is an isomorphism. So is $Dv(\omega) = S_{\perp}(\omega)^{-1} L$ on $\omega^{\perp}$: the polarization v is a local angular coordinate. This is where the equality of the state and spatial dimensions is used.

For $W \in \mathbb{C}^d_{\mathbf{k}}$ near zero, let $e(W) = (1, W)^{\mathsf{t}}$ and set

$$B_j(W) = \frac{e(W)^{\mathsf{t}} C_j e(W)}{1 + W \cdot W}.$$

The products here are bilinear. On real eigenvectors this map is the group velocity: $B(v(\xi)) = \nabla\phi(\xi)$. Its differential at zero is $2L^{\mathsf{t}}$, with image $\omega^{\perp}$. Thus for a fixed small $w_0 > 0$ the projection of B onto $\omega^{\perp}$ is a holomorphic coordinate map on a neighborhood of $|W| \le w_0$, and the matrices $[dB(W), \omega]$ and their inverses are bounded there.

We use one further complex coordinate Φ in the normal direction ω. For the imaginary time τ and spatial coordinates $P \in \mathbb{C}^n_{\mathbf{k}}$ define the fiber chart by

$$P = \tau B(W) + \Phi\omega, \qquad R^2 = \frac{2\tau}{1 + W \cdot W}, \qquad y = |R|^2 \asymp s = |\tau|. \tag{E.3}$$

Inverting the projection of P/τ onto $\omega^{\perp}$ recovers W; Φ then follows from the component along ω. At a real polarization $W = v(\xi)$ the Rayleigh quotient $\xi \cdot B(W)$ is stationary. Since $\xi \cdot B(W) = \phi(\xi)$ there, differentiating (E.3) gives the characteristic conormal

$$d\Phi = \frac{\xi \cdot dP - \phi(\xi)\, d\tau}{\xi \cdot \omega} \qquad (W = v(\xi)).$$

Let $a = Kh$, $Q = \Phi/a$, and take the spatial fiber over (T, τ) to be the torus of Section 4 in the real variables, times

$$r = |W|^2/w_0^2 + 100|Q|^2 - 1 < 0$$

in the imaginary variables. For $a < s < 4$, differentiation of (E.3) gives

$$|dW| \lesssim s^{-1}, \qquad |d\Phi| \lesssim 1, \qquad \mathrm{d}P \asymp a^2 s^{2d}\, \mathrm{d}W\ \mathrm{d}Q. \tag{E.4}$$

The derivative bounds include the time variables: solve $dP - B\, d\tau = \tau\, dB\, dW + \omega\, d\Phi$. At fixed τ the complex Jacobian matrix in (W, Q) is $[\tau\, dB, a\omega]$, which also proves the volume formula. In particular $\partial_\omega \Phi = 1$ in the spatial variables. If $F = (W/w_0, 10Q)$, Lemma 3.1 gives $H(r) = 4(dF)^* dF$. Its spatial block is $\gtrsim s^{-2} I$ and has determinant $\asymp a^{-2} s^{-2d}$. Since $|F| \le 1$, it also satisfies $G_{\mathrm{fib}}(r) G_{\mathrm{fib}}(r)^* \le H_{\mathrm{fib}}(r)$. The full Hessian has rank n. Its quaternionic null line belongs to $\ker d\Phi \cap G(r)^{\perp}$, since $d\Phi$ is a multiple of the last row of dF and $G(r) = 2(dF)^* F\mathbf{i}$. The spatial Hessian of $|\Phi|^2$ has rank one, with nonzero eigenvalue comparable to one. These are the geometric and boundary properties needed for the determinant energy.

E.2 The current and its Gaussian multiplier

Take a finite periodic Fourier sum $g_T = e^{-\mathrm{i}T\phi(D)} f$ in the annulus $|\xi| \asymp h^{-1}$ in the chosen chart, with $|v(\xi)| \le \gamma$, where $\gamma < w_0/10$ is fixed below. Write its individual terms as u_ξ. The entire vector wave $\sum_\xi |\xi|^{d/2} E(\xi) u_\xi$ solves the symmetric system with matrices $C^0 = I$, $C^j = C_j$. To apply the flux construction of Section 2, write its paired auxiliary states as $w = w_1 + \mathbf{k} w_2 = Re(W)$. The holomorphic flux map is

$$\Theta(w, \Phi) = \left(\tfrac{1}{2} w^{\mathsf{t}} C^\mu w\right)_{\mu=0}^{n} + (0, \omega)\Phi.$$

Its output is exactly (τ, P) in (E.3). Writing $M_{\mu a} = (C^\mu w)_a$, its real Jacobian is $J = |\det[M, (0, \omega)]|^2 = |R|^{2n} j(W)$ with $j(W) \asymp 1$: each of the first n columns has a factor R, and the remaining matrix is invertible by (E.3).

Let G_{sys} be the conjugate of the vector wave evaluated at $(T, X) - \mathrm{i}q_{\mathbb{R}}$, where $q^\mu_{\mathbb{R}} = w^* C^\mu w/2$. Pushing the tensor of Lemma 2.1 forward as a density by (2.7), the general row formula (2.8) gives

$$\chi_\mu = J^{-1/2} \sum_a M_{\mu a} \mathbf{i} (G_{\mathrm{sys}})_a, \qquad A^0 = \chi\chi^*. \tag{E.5}$$

The change of variables preserves column divergence: its Hessian term pairs the two equal diagonal state blocks with $(\partial_{w_{1a}w_{1b}} + \partial_{w_{2a}w_{2b}})\Theta^\mu = 0$, and the mixed state block vanishes. Since Φ is the passive coordinate, $d\Phi\, M = 0$ and hence $d\Phi\, \chi = 0$. The choice of root R affects only the sign of χ, so A^0 is single valued.

The attenuation of each plane wave is determined by $S(\xi)E(\xi) = 0$:

$$\exp(-w^* S(\xi) w/2) = e^{-\Psi}, \qquad \Psi = \frac{y}{2h}(W - v(\eta))^* S_\perp(\eta)(W - v(\eta)), \quad \eta = h\xi. \tag{E.6}$$

For each row norm at fixed imaginary coordinates we may remove the common left unit $R/|R|$ in (E.5). The remaining volume factor is $|R|/\sqrt{J} = j(W)^{-1/2} y^{-d/2}$. With respect to the orthogonal planes $\mathbb{C}_{\mathbf{i}}$ and $\mathbf{k}\mathbb{C}_{\mathbf{i}}$, each row is then a pair of scalar multipliers on g_T, up to conjugation and multiplication by $\mathbf{i}$. Their real symbols are the two components in $\mathbb{C}_{\mathbf{k}}$ of

$$j(W)^{-1/2} e(W)^{\mathsf{t}} C^\mu E(\eta)\, |\eta|^{d/2} (hy)^{-d/2} e^{-\Psi}. \tag{E.7}$$

For the time row the coefficient is $j(W)^{-1/2}(1 + W \cdot v(\eta))$, whose real part is bounded below. Thus $A^0_{TT} \geq |\mathsf{S}_0 g_T|^2$, where S_0 is the multiplier with symbol the real part of (E.7) for $\mu = 0$. It can be inverted on the cap.

E.3 The fixed cap estimate

Normalize $\|f\|_2 = 1$. Write $\sigma = s/h$, $p = p_c$, $q = p/2 = n/d$. With the kernels J_ℓ and cutoff $g(Q)$ of (4.5) put

$$A_{\mathrm{w}} = g(Q)(J_h *_X A^0), \qquad A = A_{\mathrm{w}} + h^{50n} C(H(r)), \qquad \mu = A_{TT}, \qquad B_0 = \lambda |\Phi|^2, \quad \lambda = h^{-30n}.$$

Since Φ is independent of X, the averaged current still annihilates $G(g(Q))$, so the cutoff preserves conservation. The positive semidefinite cofactor is conserved, takes values in $\ker H(r) \subset \ker d\Phi$, and has TT entry $\det H_{\mathrm{fib}}(r) > 0$. Thus $A : H(B_0) = 0$ and $\mu > 0$. The spatial background $H_{\mathrm{fib}}(B_0)$ has a single nonzero eigenvalue $\asymp \lambda$. Theorem A.1 gives the solution $u \leq 0$ of (4.8), smooth in (T, τ). The null line of $H(r)$ lies in $G(r)^\perp \cap \ker H(B_0)$, so both normal cofactor terms vanish as in Proposition 4.2. That balance and Lemma 4.3 give, for the energy $\mathcal{F} = \mathcal{J}_z(u)$ of (4.11),

$$\mathcal{F} \asymp \int_{r<0} (-u)\mu \, \mathrm{d}X \, \mathrm{d}P, \qquad \Delta_{T,\tau}\mathcal{F} \leq \int (u/r) G(r)^* A_{\mathrm{w}} G(r)\, \delta(r) \, \mathrm{d}X \, \mathrm{d}P. \tag{E.8}$$

We check the two Gaussian estimates used in the proof of Proposition 5.1, throughout $2a \leq s \leq 3$. First multiply each row symbol (E.7) by a smooth cutoff equal to one on the data, supported in a fixed enlarged annulus and in $|v| < w_0/3$ inside the chart. Since $|\partial_\eta^\alpha \Psi| \lesssim_\alpha \sigma$, the derivatives of these symbols through order $2n + 4$ are bounded by $C(hs)^{-d/2}\sigma^m$ for some m. On the active wall $r = 0$, $g(Q) \neq 0$, we have $|W| \geq \sqrt{3}\, w_0/2$. The lower bound $S_\perp(\eta) \geq cI$ gives the additional factor $e^{-c_0\sigma}$ with $c_0 > 0$ independent of γ. The kernel estimate leading to (5.5) now gives, for $P_* = (J_{Ds} * |g_T|^2)^{1/d}$ and fixed large D,

$$\mu \lesssim s^{-2d}\sigma^m P_*^d \quad (r \leq 0), \qquad \|A_{\mathrm{w}}\| \lesssim s^{-2d}\sigma^m e^{-c_0\sigma} P_*^d \quad (r = 0). \tag{E.9}$$

The cofactor adds $O(h^{50n} a^{-2} s^{-2d})$ to μ; the nearest lattice translate gives $P_*^d \gtrsim h^{n+4}$ since $\|g_T\|_2 = 1$.

For the inverse estimate take a center with $|W| \leq \gamma$, $|Q| \leq 1/400$. By (E.4), its spatial ball of radius h lies in $|W| \leq 2\gamma$, $|Q| \leq 1/100$, where $g = 1$, for K large relative to $1/\gamma$. At each imaginary point of this ball invert S_0 using a smooth cutoff equal to one on the data, supported in the same enlarged annulus and in $|v| \leq 2\gamma$. On its support $\Psi \leq C_1\gamma^2\sigma$, so the reciprocal with cutoff has

derivatives through order $2n+4$ bounded by $C_\gamma (hs)^{d/2}\sigma^m e^{C_1\gamma^2\sigma}$ with C_1 independent of γ. Kernel domination by J_h and $A^0_{TT} \geq |\mathsf{S}_0 g_T|^2$ give, as in (5.11),

$$\mu \geq m_X = c_\gamma (hs)^{-d}\sigma^{-m} e^{-2C_1\gamma^2\sigma} |g_T(X)|^2$$

on the whole ball: moving the real coordinate by at most h changes J_h only by a bounded factor.

The two comparison functions of Section 5 now apply. By (5.7), $H_{\mathrm{fib}}(rP_*) \geq P_* H_{\mathrm{fib}}(r)/2$. Since $H_{\mathrm{fib}}(r) \gtrsim s^{-2} I$ and the background has rank one, comparison with $C\sigma^{m'} r P_*$ gives $0 \leq u/r \lesssim \sigma^{m'} P_*$ by (E.9). On the ball the quadratic function $\epsilon m_X^{1/d}(\ell^2 - h^2)$, where ℓ is distance from the center, gives $(-u)\mu \gtrsim h^2 m_X^q$ at the center. This comparison uses $m_X^{1/d} \leq \lambda$: the $O(h^{-n})$ Fourier coefficients sum in modulus to $O(h^{-n/2})$ by Parseval, and (E.7) is $O(h^{-d})$. Together with the cofactor bound this gives $\mu^{1/d} \leq \lambda$ for small h.

Tangency removes $G(|Q|^2)$ from the flux; the remaining part of $G(r)$ is $O(s^{-1})$ by (E.4). The same Jacobian estimate gives imaginary volume $O(a^2 s^{2d})$ for both the fiber and its wall with weight $\delta(r)$, and volume $\asymp_\gamma a^2 s^{2d}$ for the centers. Using $\int P_*^{d+1}\,\mathrm{d}X \leq \|g_T\|_p^p$ and $h^2 s^{2d}(hs)^{-dq} = \sigma^{n-2}$ in (E.8), we get

$$c_\gamma a^2 \sigma^{-m} e^{-2qC_1\gamma^2\sigma} \|g_T\|_p^p \leq \mathcal{F} \leq C a^2 \sigma^m \|g_T\|_p^p, \tag{E.10}$$

$$\Delta_{T,\tau}\mathcal{F} \leq C a^2 s^{-2} \sigma^m e^{-c_0\sigma} \|g_T\|_p^p. \tag{E.11}$$

Choose γ so that $2qC_1\gamma^2 < c_0/2$, then fix K. Division gives $\Delta\mathcal{F} \leq C s^{-2} e^{-cs/h}\mathcal{F}$. At $T = 0$ the upper bound on $s = 4a$ uses $g_0 = f$; on $1 \leq T \leq 2$ the lower bound on $s = 2a$ controls $\|g_T\|_p^p$. The values of σ on these circles are fixed. As in the proof of Proposition 6.3, Lemma 6.1 with $b = 2a$ and $r = 4a$ gives

$$\int_1^2 \|g_T\|_p^p \,\mathrm{d}T \lesssim [\log(2/h)]^2 \|f\|_p^p.$$

Homogeneity removes the normalization.

Finitely many such caps cover the directions in $\operatorname{supp} b$; their smooth annular cutoffs have uniformly integrable kernels. To remove the period, choose $M \geq 1$ larger than $\sup_{\operatorname{supp} b} |\nabla\phi|$ and apply the torus estimate on the same lattice to C/M and $b_M(\eta) = b(\eta/M)$. By homogeneity $|\nabla(\phi/M)| \leq 1$ on $\operatorname{supp} b_M$. For $|X| \geq 3$ the kernel phase has gradient $X - (T/M)\nabla\phi(\eta)$ of size $\gtrsim |X|$, which gives the decay (7.2). The cube argument of Section 7 gives the Euclidean estimate for $(C/M, b_M)$. For $f_M(X) = f(MX)$ one has

$$e^{-\mathrm{i}T\phi(D)/M} b_M(hD) f_M(X) = (e^{-\mathrm{i}T\phi(D)} b(hD) f)(MX).$$

Both norms gain the same factor $M^{-n/p}$, proving (E.1) for small h. Smooth multiplier estimates cover the remaining compact range. Applying the result to $-C$ with cutoff $\bar{b}$ and conjugating with $f(X) \mapsto \overline{f(-X)}$ gives the smallest-eigenvalue assertion.

E.4 A lower bound on a single patch

For every nonzero b in Theorem E.1, the logarithmic power in (E.1) must be at least $1/p_c$. Here is the local version of the focusing test in Proposition 8.1. Rotate a neighborhood where $b \neq 0$ into the coordinates $\eta = \rho(\alpha, 1)$, $\rho > 0$, and write $\phi(\eta) = \rho\varphi(\alpha)$. The matrix $D^2\varphi$ is positive definite there, since the Hessian of ϕ has only the radial kernel. Take $h = N^{-1}$ and $\widehat{f}_N(\xi) = e^{(3/2)\mathrm{i}\phi(\xi)}(\psi/b)(\xi/N)$, where ψ is a nonnegative smooth bump supported in this neighborhood and strictly positive on a smaller closed product patch. At the initial time, stationary phase in α gives $\|f_N\|_\infty \lesssim N^{n-d/2}$ uniformly in X, and Plancherel gives $\|f_N\|_2 \lesssim N^{n/2}$. Thus $\|f_N\|_p \lesssim N^{(n+1)/2-1/p}$ for $p \geq 2$.

At time $3/2+\delta$, set $X' = \delta\nabla\varphi(\alpha)$ and $X_n = \delta(\varphi(\alpha) - \alpha\cdot\nabla\varphi(\alpha)) + z/N$. For α in the smaller patch and $|z| \le c$, this set has volume $\asymp \delta^d/N$. Its angular stationary point is α. For $N\delta$ large, stationary phase now gives output of modulus at least $cN^{n-d/2}\delta^{-d/2}$: up to a nonzero smooth factor and constant phase the leading radial integral is $\int e^{\mathrm{i}\rho z}\rho^{d/2}\psi(\rho(\alpha,1))\,\mathrm{d}\rho$, uniformly nonzero for small c. The angular remainder is smaller by $O((N\delta)^{-1})$ [74, Chapter VIII]. The pth power of the ratio of the output and input norms thus dominates $\int_{C/N}^{1/4}\delta^{d-dp/2}\,\mathrm{d}\delta$. For $p = p_c$ this is comparable to $\log N$.

References

[1] Per Åhag, Urban Cegrell, Sławomir Kołodziej, H. H. Phậm, and Ahmed Zeriahi. Partial pluricomplex energy and integrability exponents of plurisubharmonic functions. *Advances in Mathematics*, 222(6):2036–2058, 2009. doi: 10.1016/j.aim.2009.07.002.

[2] Semyon Alesker. Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables. *Bulletin des Sciences Mathématiques*, 127(1):1–35, 2003. doi: 10.1016/S0007-4497(02)00004-0.

[3] Semyon Alesker. Quaternionic Monge–Ampère equations. *Journal of Geometric Analysis*, 13 (2):205–238, 2003. doi: 10.1007/BF02930695.

[4] Semyon Alesker. Solvability of the quaternionic Monge–Ampère equation on compact manifolds with a flat hyperKähler metric. *Advances in Mathematics*, 241:192–219, 2013. doi: 10.1016/j.aim.2013.03.021.

[5] Thierry Aubin. Réduction du cas positif de l'équation de Monge–Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité. *Journal of Functional Analysis*, 57 (2):143–153, 1984. doi: 10.1016/0022-1236(84)90093-4.

[6] M. S. Baouendi and F. Trèves. A microlocal version of Bochner's tube theorem. *Indiana University Mathematics Journal*, 31(6):885–895, 1982. doi: 10.1512/iumj.1982.31.31060.

[7] William Beckner, Anthony Carbery, Stephen Semmes, and Fernando Soria. A note on restriction of the Fourier transform to spheres. *Bulletin of the London Mathematical Society*, 21(4):394–398, 1989. doi: 10.1112/blms/21.4.394.

[8] David Beltran, Jonathan Hickman, and Christopher D. Sogge. Sharp local smoothing estimates for Fourier integral operators. In Paolo Ciatti and Alessio Martini, editors, *Geometric Aspects of Harmonic Analysis*, volume 45 of *Springer INdAM Series*, pages 29–105. Springer, Cham, 2021. doi: 10.1007/978-3-030-72058-2_2.

[9] David Beltran, Joris Roos, Alex Rutar, and Andreas Seeger. A fractal local smoothing problem for the wave equation. *Bulletin of the London Mathematical Society*, 57(12):3667–3690, 2025. doi: 10.1112/blms.70214.

[10] Robert J. Berman, Sébastien Boucksom, Vincent Guedj, and Ahmed Zeriahi. A variational approach to complex Monge–Ampère equations. *Publications Mathématiques de l'IHÉS*, 117: 179–245, 2013. doi: 10.1007/s10240-012-0046-6.

[11] Aline Bonami and Jean-Louis Clerc. Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques. *Transactions of the American Mathematical Society*, 183:223–263, 1973. doi: 10.1090/S0002-9947-1973-0338697-5.

[12] Jean Bourgain. Averages in the plane over convex curves and maximal operators. *Journal d'Analyse Mathématique*, 47:69–85, 1986. doi: 10.1007/BF02792533.

[13] Jean Bourgain. Besicovitch type maximal operators and applications to Fourier analysis. *Geometric and Functional Analysis*, 1(2):147–187, 1991. doi: 10.1007/BF01896376.

[14] Jean Bourgain and Ciprian Demeter. The proof of the ℓ^2 decoupling conjecture. *Annals of Mathematics*, 182(1):351–389, 2015. doi: 10.4007/annals.2015.182.1.9.

[15] Luis Caffarelli, Louis Nirenberg, and Joel Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge–Ampère equation. *Communications on Pure and Applied Mathematics*, 37(3):369–402, 1984. doi: 10.1002/cpa.3160370306.

[16] Anthony Carbery. The boundedness of the maximal Bochner–Riesz operator on $L^4(\mathbb{R}^2)$. *Duke Mathematical Journal*, 50(2):409–416, 1983. doi: 10.1215/S0012-7094-83-05018-4.

[17] Anthony Carbery, José L. Rubio de Francia, and Luis Vega. Almost everywhere summability of Fourier integrals. *Journal of the London Mathematical Society*, 38(3):513–524, 1988. doi: 10.1112/jlms/s2-38.3.513.

[18] Leonardo Colzani, Alice Cominardi, and Krzysztof Stempak. Radial solutions to the wave equation. *Annali di Matematica Pura ed Applicata*, 181(1):25–54, 2002. doi: 10.1007/s102310100025.

[19] Antonio Córdoba. The Kakeya maximal function and the spherical summation multipliers. *American Journal of Mathematics*, 99(1):1–22, 1977. doi: 10.2307/2374006.

[20] Adam Cushman, Ciprian Demeter, and Shukun Wu. Diameter-free reverse inequalities and superorthogonality. Preprint, arXiv:2609.13105, 2026. URL `https://arxiv.org/abs/2609.13105`.

[21] Roy O. Davies. Some remarks on the Kakeya problem. *Proceedings of the Cambridge Philosophical Society*, 69(3):417–421, 1971. doi: 10.1017/S0305004100046867.

[22] Lawrence C. Evans. Classical solutions of fully nonlinear, convex, second-order elliptic equations. *Communications on Pure and Applied Mathematics*, 35(3):333–363, 1982. doi: 10.1002/cpa.3160350303.

[23] Charles Fefferman. The multiplier problem for the ball. *Annals of Mathematics*, 94(2):330–336, 1971. doi: 10.2307/1970864.

[24] Rafael J. Fernández-Delgado and Mikael de la Salle. A construction of Kakeya sets in arbitrary dimension. Preprint, arXiv:2607.14824v3, 2026.

[25] Kurt O. Friedrichs. Symmetric hyperbolic linear differential equations. *Communications on Pure and Applied Mathematics*, 7(2):345–392, 1954. doi: 10.1002/cpa.3160070206.

[26] Shengwen Gan, Yifan Jing, and Shukun Wu. New bounds for Stein's square function in $\mathbb{R}^3$. In *Recent Developments in Harmonic Analysis and its Applications*, volume 792 of *Contemporary Mathematics*, pages 19–65. American Mathematical Society, Providence, RI, 2024. doi: 10.1090/conm/792/15893.

[27] Shengwen Gan, Changkeun Oh, and Shukun Wu. New bounds for Stein's square functions in higher dimensions. *Advances in Mathematics*, 475:110342, 2025. doi: 10.1016/j.aim.2025.110342.

[28] Shengwen Gan, Danqing He, Xiaochun Li, and Shukun Wu. On local smoothing estimates for wave equations. Preprint, arXiv:2502.05973v2, 2026. URL https://arxiv.org/abs/2502.05973v2. Revised January 4, 2026.

[29] Chuanwei Gao, Bochen Liu, Changxing Miao, and Yakun Xi. Improved local smoothing estimate for the wave equation in higher dimensions. *Journal of Functional Analysis*, 284(9): 109879, 2023. doi: 10.1016/j.jfa.2023.109879.

[30] Chuanwei Gao, Diankun Liu, and Yakun Xi. Curved Kakeya sets and Nikodym problems on manifolds. *Peking Mathematical Journal*, 2026. doi: 10.1007/s42543-025-00114-1. Published online February 2, 2026.

[31] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer, Berlin, second edition, 2001. doi: 10.1007/978-3-642-61798-0. Reprint of the revised second edition; originally published in Grundlehren der mathematischen Wissenschaften, vol. 224.

[32] Loukas Grafakos. *Classical Fourier Analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, third edition, 2014. doi: 10.1007/978-1-4939-1194-3.

[33] Loukas Grafakos, Liguang Liu, and Dachun Yang. Vector-valued singular integrals and maximal functions on spaces of homogeneous type. *Mathematica Scandinavica*, 104(2):296–310, 2009. doi: 10.7146/math.scand.a-15099.

[34] Shaoming Guo, Joris Roos, and Po-Lam Yung. Sharp variation-norm estimates for oscillatory integrals related to Carleson's theorem. *Analysis & PDE*, 13(5):1457–1500, 2020. doi: 10.2140/apde.2020.13.1457.

[35] Larry Guth, Hong Wang, and Ruixiang Zhang. A sharp square function estimate for the cone in $\mathbb{R}^3$. *Annals of Mathematics*, 192(2):551–581, 2020. doi: 10.4007/annals.2020.192.2.6.

[36] Larry Guth, Hong Wang, and Joshua Zahl. A streamlined proof of the Kakeya set conjecture in $\mathbb{R}^3$. Preprint, arXiv:2601.14411v2, 2026. Version 2, October 11, 2026.

[37] Yaryong Heo, Fëdor Nazarov, and Andreas Seeger. Radial Fourier multipliers in high dimensions. *Acta Mathematica*, 206(1):55–92, 2011. doi: 10.1007/s11511-011-0059-x.

[38] Lars Hörmander. Oscillatory integrals and multipliers on FL^p. *Arkiv för Matematik*, 11:1–11, 1973. doi: 10.1007/BF02388505.

[39] Kôdi Husimi. Some formal properties of the density matrix. *Proceedings of the Physico-Mathematical Society of Japan, Third Series*, 22(4):264–314, 1940. doi: 10.11429/ppmsj1919.22.4_264.

[40] Roger L. Jones, Andreas Seeger, and James Wright. Strong variational and jump inequalities in harmonic analysis. *Transactions of the American Mathematical Society*, 360(12):6711–6742, 2008. doi: 10.1090/S0002-9947-08-04538-8.

[41] Gerald Kaiser. *Quantum Physics, Relativity, and Complex Spacetime: Towards a New Synthesis*, volume 163 of *North-Holland Mathematics Studies*. North-Holland, Amsterdam, 1990. URL https://arxiv.org/abs/0910.0352.

[42] Gerald Kaiser and Raymond F. Streater. Windowed Radon transforms, analytic signals, and the wave equation. In Charles K. Chui, editor, *Wavelets: A Tutorial in Theory and Applications*, volume 2 of *Wavelet Analysis and Its Applications*, pages 399–441. Academic Press, Boston, 1992. URL https://arxiv.org/abs/math-ph/0108010.

[43] Tosio Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer, Berlin, second edition, 1995. doi: 10.1007/978-3-642-66282-9.

[44] Uri Keich. On L^p bounds for Kakeya maximal functions and the Minkowski dimension in $\mathbb{R}^2$. *Bulletin of the London Mathematical Society*, 31(2):213–221, 1999. doi: 10.1112/S0024609398005372.

[45] Johannes Keller, Caroline Lasser, and Tomoki Ohsawa. A new phase space density for quantum expectations. *SIAM Journal on Mathematical Analysis*, 48(1):513–537, 2016. doi: 10.1137/15M1028388.

[46] Jongchon Kim. Some remarks on Fourier restriction estimates. Preprint, arXiv:1702.01231, 2017.

[47] Nikolai V. Krylov. Boundedly nonhomogeneous elliptic and parabolic equations. *Mathematics of the USSR-Izvestiya*, 20(3):459–492, 1983. doi: 10.1070/IM1983v020n03ABEH001360. Russian original: Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), no. 3, 487–523.

[48] Nikolai V. Krylov. Boundedly nonhomogeneous elliptic and parabolic equations in a domain. *Mathematics of the USSR-Izvestiya*, 22(1):67–97, 1984. doi: 10.1070/IM1984v022n01ABEH001434. Russian original: Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), no. 1, 75–108.

[49] Izabella Łaba and Thomas Wolff. A local smoothing estimate in higher dimensions. *Journal d'Analyse Mathématique*, 88:149–171, 2002. doi: 10.1007/BF02786576.

[50] Sanghyuk Lee, Keith M. Rogers, and Andreas Seeger. Improved bounds for Stein's square functions. *Proceedings of the London Mathematical Society*, 104(6):1198–1234, 2012. doi: 10.1112/plms/pdr067.

[51] Sanghyuk Lee, Keith M. Rogers, and Andreas Seeger. On space-time estimates for the Schrödinger operator. *Journal de Mathématiques Pures et Appliquées*, 99(1):62–85, 2013. doi: 10.1016/j.matpur.2012.06.003.

[52] Toshiki Mabuchi. K-energy maps integrating Futaki invariants. *Tohoku Mathematical Journal*, 38(4):575–593, 1986. doi: 10.2748/tmj/1178228410.

[53] William P. Minicozzi, II and Christopher D. Sogge. Negative results for Nikodym maximal functions and related oscillatory integrals in curved space. *Mathematical Research Letters*, 4 (2–3):221–237, 1997. doi: 10.4310/MRL.1997.v4.n2.a5.

[54] Akihiko Miyachi. On some estimates for the wave equation in L^p and H^p. *Journal of the Faculty of Science, University of Tokyo, Section IA, Mathematics*, 27(2):331–354, 1980.

[55] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Wave front sets, local smoothing and Bourgain's circular maximal theorem. *Annals of Mathematics*, 136(1):207–218, 1992. doi: 10.2307/2946549.

[56] Gerd Mockenhaupt, Andreas Seeger, and Christopher D. Sogge. Local smoothing of Fourier integral operators and Carleson–Sjölin estimates. *Journal of the American Mathematical Society*, 6(1):65–130, 1993. doi: 10.1090/S0894-0347-1993-1168960-6.

[57] Peter Mörters and Yuval Peres. *Brownian Motion*, volume 30 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2010. doi: 10.1017/CBO9780511750489.

[58] Detlef Müller and Andreas Seeger. Inequalities for spherically symmetric solutions of the wave equation. *Mathematische Zeitschrift*, 218:417–426, 1995. doi: 10.1007/BF02571912.

[59] Arian Nadjimzadah. Lifting curved Kakeya sets to linear Kakeya sets. Preprint, arXiv:2609.06299v2, 2026. Revised October 20, 2026.

[60] F. J. Narcowich, P. Petrushev, and J. D. Ward. Localized tight frames on spheres. *SIAM Journal on Mathematical Analysis*, 38(2):574–594, 2006. doi: 10.1137/040614359.

[61] Fabio Nicola. Slicing surfaces and the Fourier restriction conjecture. *Proceedings of the Edinburgh Mathematical Society*, 52(2):515–527, 2009. doi: 10.1017/S0013091507000995.

[62] OpenAI. Every Four-Dimensional Kakeya Set Has Full Hausdorff Dimension. OpenAI Math Release preprint OAI:Every-four-dimensional-Kakeya-set-has-full-Hausdorff-dimension-September-24-2026, 2026.

[63] OpenAI. The Kakeya maximal conjecture in three dimensions. OpenAI Math Release preprint OAI:The-Kakeya-maximal-conjecture-in-three-dimensions-September-23-2026, 2026.

[64] OpenAI. Elliptic capacity propagation and Fourier restriction to the sphere. OpenAI Math Release preprint OAI:Elliptic-capacity-propagation-and-Fourier-restriction-to-the-sphere-September-24-2026, 2026.

[65] Juan C. Peral. L^p estimates for the wave equation. *Journal of Functional Analysis*, 36(1): 114–145, 1980. doi: 10.1016/0022-1236(80)90110-X.

[66] Keith M. Rogers. A local smoothing estimate for the Schrödinger equation. *Advances in Mathematics*, 219(6):2105–2122, 2008. doi: 10.1016/j.aim.2008.08.008.

[67] Andreas Seeger. Some inequalities for singular convolution operators in L^p-spaces. *Transactions of the American Mathematical Society*, 308(1):259–272, 1988. doi: 10.1090/S0002-9947-1988-0955772-3.

[68] Andreas Seeger, Christopher D. Sogge, and Elias M. Stein. Regularity properties of Fourier integral operators. *Annals of Mathematics*, 134(2):231–251, 1991. doi: 10.2307/2944346.

[69] Denis Serre. Divergence-free positive symmetric tensors and fluid dynamics. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(5):1209–1234, 2018. doi: 10.1016/j.anihpc.2017.11.002.

[70] Denis Serre. Compensated integrability. Applications to the Vlasov–Poisson equation and other models in mathematical physics. *Journal de Mathématiques Pures et Appliquées*, 127:67–88, 2019. doi: 10.1016/j.matpur.2018.06.025.

[71] Denis Serre. Mixed determinants, compensated integrability, and new a priori estimates in gas dynamics. *Quarterly of Applied Mathematics*, 81(2):281–295, 2023. doi: 10.1090/qam/1640.

[72] Christopher D. Sogge. Propagation of singularities and maximal functions in the plane. *Inventiones Mathematicae*, 104(2):349–376, 1991. doi: 10.1007/BF01245080.

[73] Elias M. Stein. Localization and summability of multiple Fourier series. *Acta Mathematica*, 100:93–147, 1958. doi: 10.1007/BF02559603.

[74] Elias M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. doi: 10.1515/9781400883929. With the assistance of Timothy S. Murphy.

[75] Terence Tao. The Bochner–Riesz conjecture implies the restriction conjecture. *Duke Mathematical Journal*, 96(2):363–375, 1999. doi: 10.1215/S0012-7094-99-09610-2.

[76] Dongrui Wan. A variational approach to the quaternionic Monge–Ampère equation. *Annali di Matematica Pura ed Applicata*, 199(6):2125–2150, 2020. doi: 10.1007/s10231-020-00960-z.

[77] Hong Wang and Joshua Zahl. Volume estimates for unions of convex sets, and the Kakeya set conjecture in three dimensions. Preprint, arXiv:2502.17655, 2025.

[78] Xiangyu Wang. Wave packet density in local smoothing: refinements and limitations. Preprint, arXiv:2609.34675v4, 2026. URL `https://arxiv.org/abs/2609.34675v4`. Revised October 16, 2026.

[79] Thomas Wolff. Recent work connected with the Kakeya problem. In Hugo Rossi, editor, *Prospects in Mathematics (Princeton, NJ, 1996)*, pages 129–162. American Mathematical Society, Providence, RI, 1999.

[80] Thomas Wolff. Local smoothing type estimates on L^p for large p. *Geometric and Functional Analysis*, 10(5):1237–1288, 2000. doi: 10.1007/PL00001652.

[81] Jingyong Zhu. Dirichlet problem of quaternionic Monge–Ampère equations. *Israel Journal of Mathematics*, 214(2):597–619, 2016. doi: 10.1007/s11856-016-1329-2.